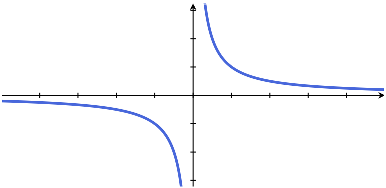
,

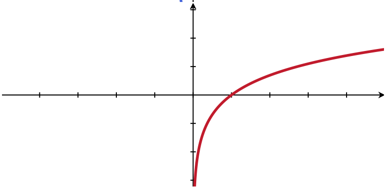
Exponential Growth and Decay

Remember: Antiderivative of $1/x$

$1/x$:



$\ln(x)$:



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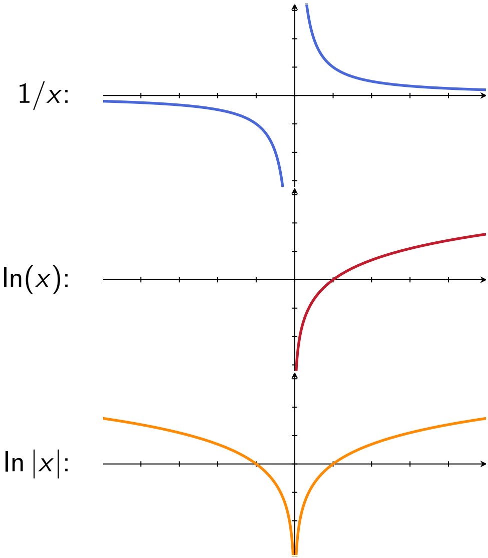

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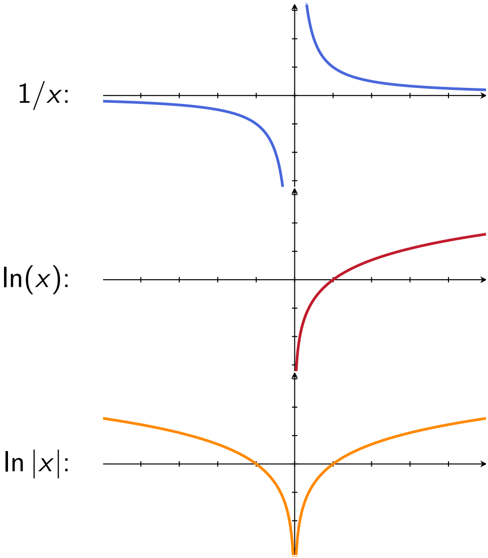

So

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

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The plan:

1. Put it into math, i.e. Write down an initial value problem.

1'. Look at slope fields to make sure the IVP makes sense.

2. Find the general solution to the IVP.
3. Plug in the points and find the particular solution.
4. Calculate the value of the solution when $t = 24$.

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Step 1: Put into math. Initial value problem:

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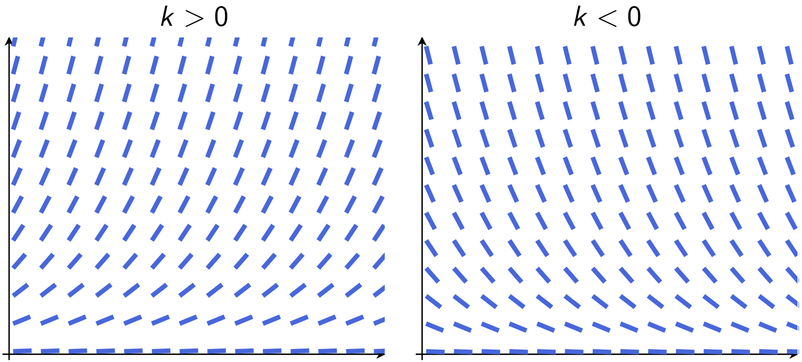

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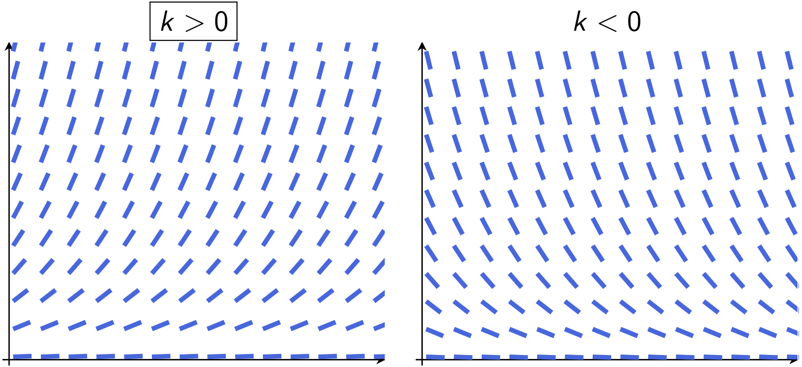

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Putting it together:

$$\ln|y| = kt + C \implies |y| = e^{kt+C} = e^C * e^{kt}$$

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$$700 = y(0) = Ae^0 = A, \quad \text{so } y = 700e^{kt}$$

$$900 = 700e^{12k} \implies 12k = \ln(900/700) = ln(9/7)$$
$$\implies k = \frac{1}{12}\ln(9/7) \approx \boxed{0.021}$$

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Particular solution: $\boxed{y = 700e^{t*\frac{1}{12}\ln(9/7)}}$

Note: another way to write this is

$$y = 700e^{t*\frac{1}{12}\ln(9/7)} = 700\left(e^{ln(9/7)}\right)^{t/12} = 700\left(\frac{9}{7}\right)^{t/12}$$

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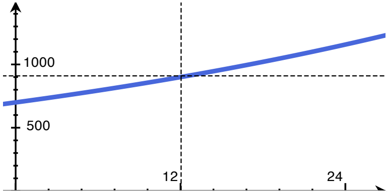

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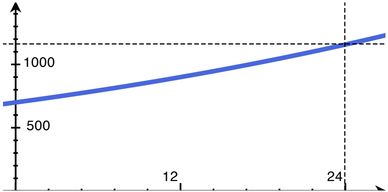

$$y(24) = 700\left(\frac{9}{7}\right)^{24/12} = 700\left(\frac{9}{7}\right)^{2} = 8100/7 \approx 1157.14$$

Example 2: Objects heat or cool at a rate proportional to the difference between their temperature and the ambient temperature. Suppose a pie is pulled out of the oven (heated to 370°F), and put into a room that's 70°F. After 10 minutes, the center of the pie is 340°F.

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Step 1: Put into math. Initial value problem:

$$\frac{dy}{dt} = k(y - 70), \qquad y(0) = 370, \quad y(10) = 340$$

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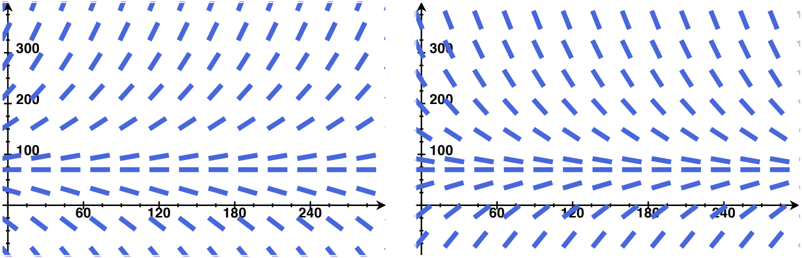

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$$\boxed{y = Ae^{kt} + 70}$$

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General solution: $\boxed{y = Ae^{kt} + 70}$

What do we expect from k and A?

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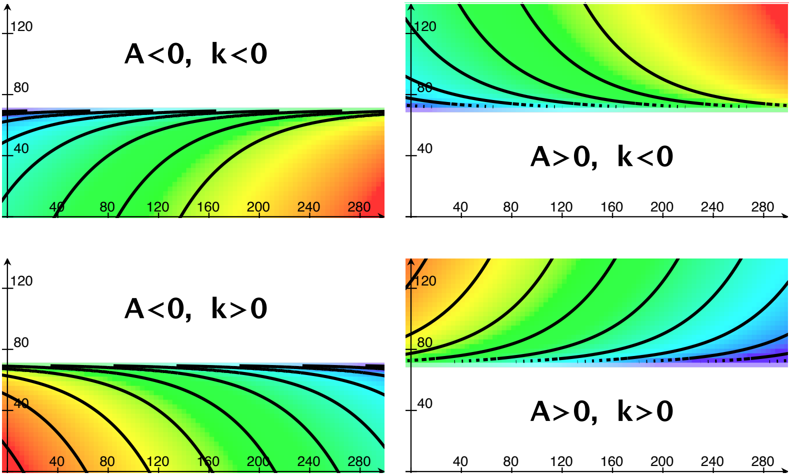

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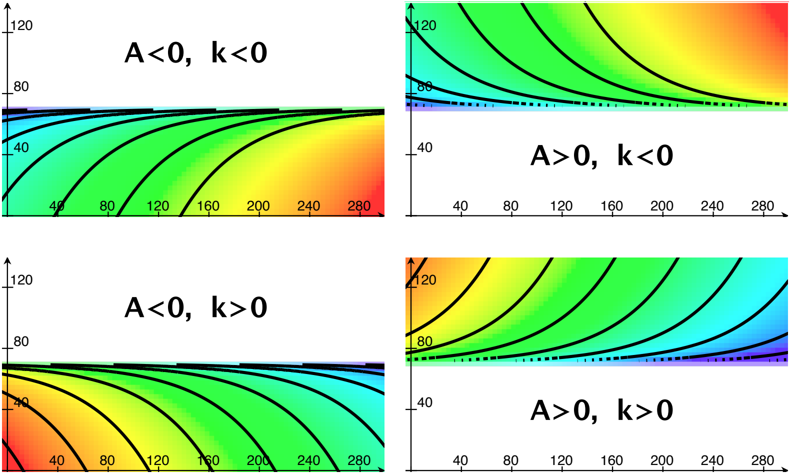
What do we expect from k and A? $A > 0,\ k < 0$



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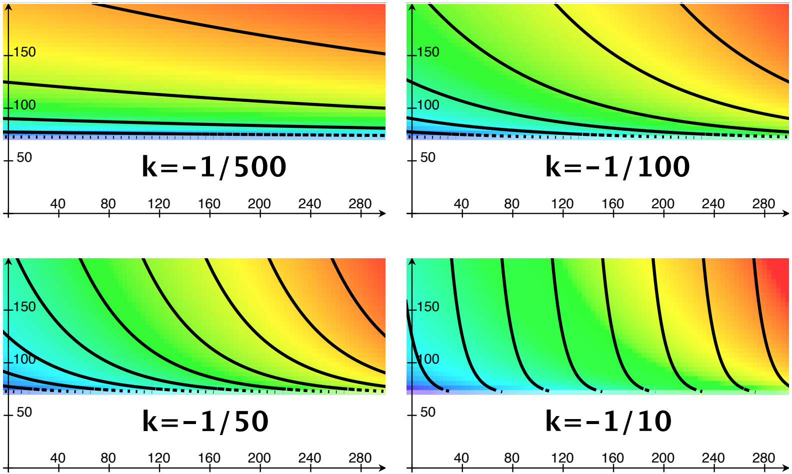

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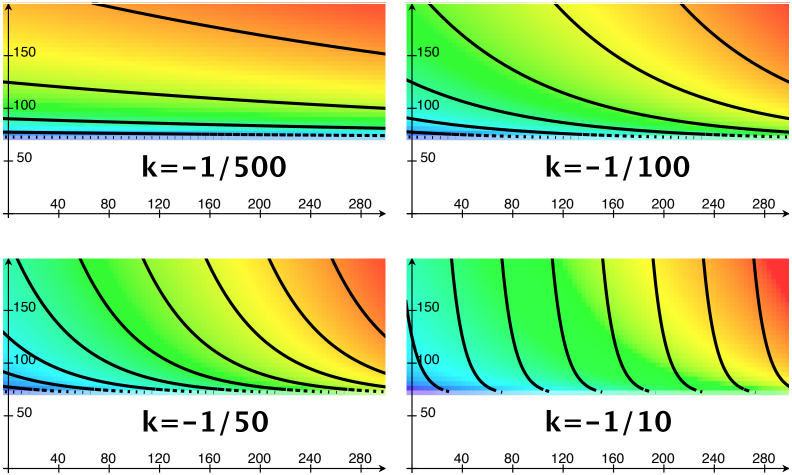
What do we expect from k and A? $A > 0$, $k < 0$, and k small



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Step 3: Plug in points and find particular solution

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Step 3: Plug in points and find particular solution

$$370 = y(0) = Ae^0 + 70, \qquad \text{so } \boxed{A = 300}$$

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$$370 = y(0) = Ae^0 + 70, \qquad \text{so } \boxed{A = 300}$$

$$340 = y(10) = 300e^{k*10} + 70$$

,

$$\text{so } k = \frac{1}{10} \ln\left(\frac{350-70}{300}\right) = \boxed{\ln(.9)/10 \approx -0.0105}.$$

$$\boxed{\text{IVP: } \frac{dy}{dt} = k(y-70), \qquad y(0)=370, \quad y(10)=340}$$

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What do we expect from k and A? $A > 0$, $k < 0$, and k small

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So the particular solution is

$$\boxed{y = 300e^{t*\ln(.9)/10} + 70}$$

Example 2: Objects heat or cool at a rate proportional to the difference between their temperature and the ambient temperature. Suppose a pie is pulled out of the oven (heated to 370°F), and put into a room that's 70°F. After 10 minutes, the center of the pie is 340°F.

(a) How hot is the pie after 20 minutes?

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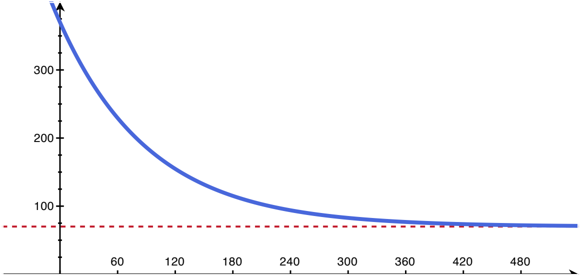

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Answers:

(a) $y(20) = 300e^{20*\ln(.9)/10} + 70 = \boxed{313}$

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Answers:

(a) $y(20) = 300e^{20*\ln(.9)/10} + 70 = \boxed{313}$

(b) $100 = 300e^{t*\ln(.9)/10} + 70$

So $e^{t*\ln(.9)/10} = 30/300 = 1/10$,

Example 2: Objects heat or cool at a rate proportional to the difference between their temperature and the ambient temperature. Suppose a pie is pulled out of the oven (heated to 370°F), and put into a room that's 70°F. After 10 minutes, the center of the pie is 340°F.

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Answers:

(a) $y(20) = 300e^{20*\ln(.9)/10} + 70 = \boxed{313}$

(b) $100 = 300e^{t*\ln(.9)/10} + 70$

So $e^{t*\ln(.9)/10} = 30/300 = 1/10$, and so

$$t = \frac{10}{\ln(.9)}\ln(.1) \approx \boxed{218.543}$$

Example 3: The isotope thorium-239 decays at a rate proportional to the amount present, and has a half-life of 24.1 days. How long does 10 grams of thorium-234 take to decay to 1 gram?

"Half-life": The time it takes for an amount of stuff to halve in size.

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IVP: $\dfrac{dy}{dt} = ky,$

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Question: What is t when $y(t) = 1$?

Example 3: The isotope thorium-239 decays at a rate proportional to the amount present, and has a half-life of 24.1 days. How long does 10 grams of thorium-234 take to decay to 1 gram?

"Half-life": The time it takes for an amount of stuff to halve in size.

$$\text{IVP:} \quad \frac{dy}{dt} = ky, \qquad y(0) = 10, \quad y(24.1) = \frac{1}{2}10.$$

Question: What is t when $y(t) = 1$?

To do:

Separate to get general solution;
Plug in points to get specific solution;
Solve $y(t) = 1$ for t

Example 4: The *logistic equation* (another model of population growth)

"A population of rabbits will grow at a rate proportional to the product of the number y of rabbits present and the difference between the population size and the carrying capacity C of their environment."

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Solving $\frac{dy}{dt} = ky(y - C)$:

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$$\frac{1}{y(y-C)} = \frac{\frac{1}{C}}{y} + \frac{-\frac{1}{C}}{y-C}$$

Solving $\frac{dy}{dt} = ky(y - C)$:

Separate and integrate:

$$\boxed{\frac{1}{C}\int \frac{1}{y}\ dy - \frac{1}{C}\int \frac{1}{y-C}\ dy = \int \frac{1}{y(y-C)}\ dy = \int k\ dt}$$

Goal: Split $\frac{1}{y(y-C)}$ into the sum of two fractions, each of which we know how to integrate: Find a and b so that

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Example 4(b):

"Suppose a population of rabbits lives in an environment that will support roughly 100 rabbits at a time. Using the logistic model, how do you expect the rabbit population to grow if you start with 10 rabbits and 6 months later you have 30 rabbits?"'

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"Suppose a population of rabbits lives in an environment that will support roughly 100 rabbits at a time. Using the logistic model, how do you expect the rabbit population to grow if you start with 10 rabbits and 6 months later you have 30 rabbits?"'

$$\frac{dy}{dt} = ky(y - 100), \qquad y \geq 0, \quad y(0) = 10, \quad y(0.5) = 30.$$

Example 4(b):

"Suppose a population of rabbits lives in an environment that will support roughly 100 rabbits at a time. Using the logistic model, how do you expect the rabbit population to grow if you start with 10 rabbits and 6 months later you have 30 rabbits?"'

$$\frac{dy}{dt} = ky(y - 100), \qquad y \geq 0, \quad y(0) = 10, \quad y(0.5) = 30.$$

Step 1: Calculate

$$\frac{1}{100}\int \frac{1}{y}\ dy - \frac{1}{100}\int \frac{1}{y-100}\ dy = \int k\ dt$$

Step 2: Solve for y.

Hint: simplify using (I) $A \ln|B| = \ln|B^A|$,

(II) $\ln|A| - \ln|B| = \ln|A/B|$, and

(III) $\ln|A| = B$ means $A = \pm e^B$.

Step 3: Plug in $y(0) = 10$ and then $y(0.5) = 30$ to solve for the integration constant and k.