Modeling with Differential Equations: Introduction to the Issues

Warm-up

Do you know a function…

…whose first derivative is the same as the function itself, i.e.

$$\frac{d}{dx} f(x) = f(x)?$$

…whose first derivative is negative of the function, i.e.

$$\frac{d}{dx} f(x) = -f(x)?$$

…whose second derivative is negative of itself, i.e.

$$\frac{d^2}{dx^2} f(x) = -f(x)?$$

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$$e^x, \qquad 2e^x, \qquad Ae^x, \qquad 0$$

...whose first derivative is negative of the function, i.e.

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$$e^{-x}, \qquad 2e^{-x}, \qquad Ae^{-x}, \qquad 0$$

...whose second derivative is negative of itself, i.e.

$$\frac{d^2}{dx^2} f(x) = -f(x)?$$

$$\cos(x), \qquad \sin(x), \qquad \sin(x) + \cos(x), \qquad A\cos(x) + B\sin(x), \qquad 0$$

Goal:
Given an equation relating a variable (e.g. x), a function (e.g. y), and its derivatives $(y', y'', \dots)$, **what is y?**
i.e. How do I solve for y?

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Why?
Many physical and biological systems can be modeled with differential equations. Also, it can be a lot harder to model a function long term than it is to measure how something changes as the system goes from one state to another.

Some examples

Obvervation: The rate of increase of a bacterial culture is proportional to the number of bacteria present at that time.

Obvervation: The motion of a mass on a spring is given by two opposing forces: (1) the force exerted by the mass in motion ($F = ma = m\frac{d^2}{dt^2}D$) and (2) the force exerted by the spring, proportional to the displacement from equilibrium ($F = kD$).

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Equation: $m * \dfrac{d^2}{dt^2}D = -kD$

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Equation: $\frac{dP}{dt} = kP$

Solution: $P = Ae^{kt}$, where A is a constant.

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Equation: $m * \frac{d^2}{dt^2}D = -kD$

Solution: $D(t) = A\cos(t * \sqrt{k/m}) + B\sin(t * \sqrt{k/m})$, where A, B, k, and m are all constants.

Slope Fields

If you can write your differential equation like

$$\frac{dy}{dx} = F(x, y)$$

then you really have a way of saying

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then I should move from here with slope $F(a, b)$."

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Some non-examples:

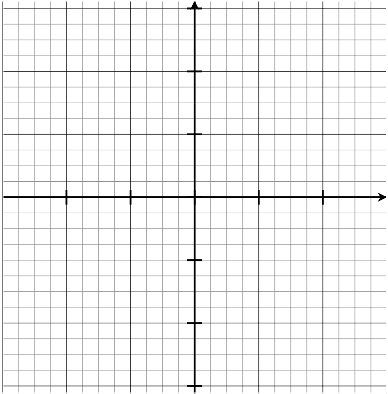
$$\frac{dy}{dx} = -\frac{x}{y} + \frac{d^2y}{dx^2}$$

$$\frac{dP}{dt} * \frac{d^2P}{dt^2} = kP$$

$$m * \frac{d^2D}{dt^2} = -kD$$

x	y	$\frac{dy}{dx} = -x/y$
0	1	
0	-1	
1	1	
1	-1	
-1	1	
-1	-1	
2	1	
1	2	
-2	0	

Slope field:



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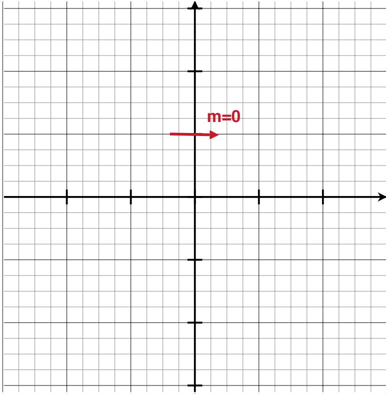

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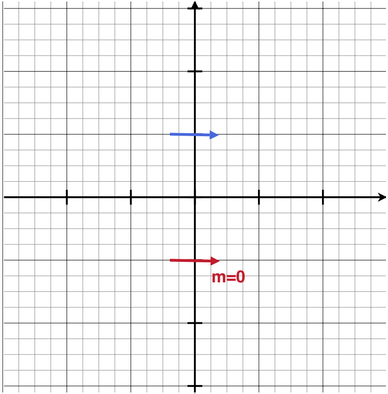

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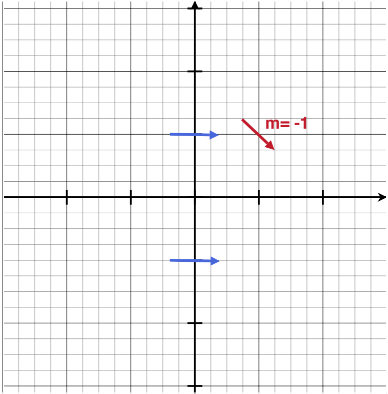

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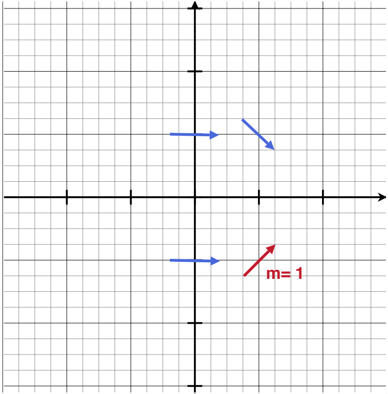

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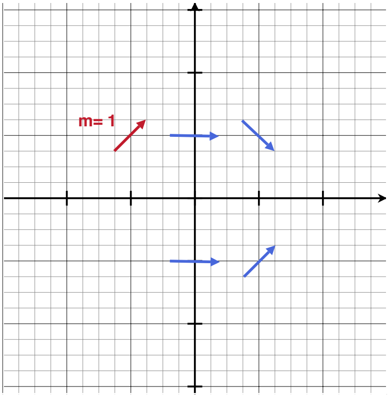

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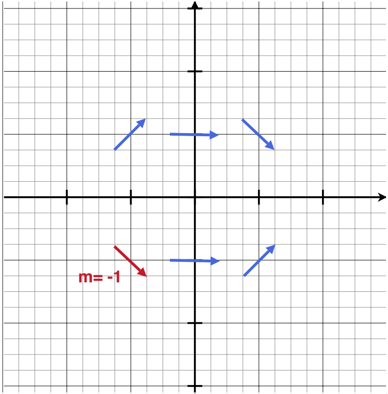

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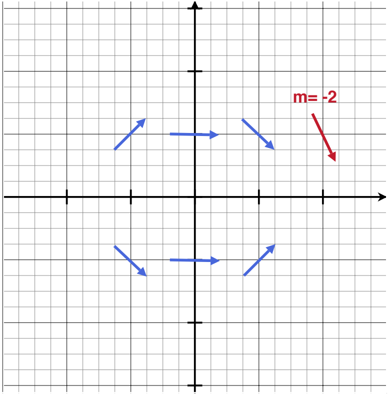

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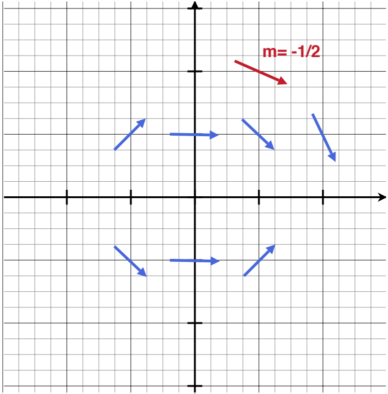

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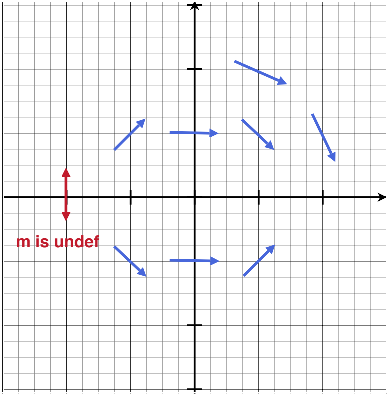

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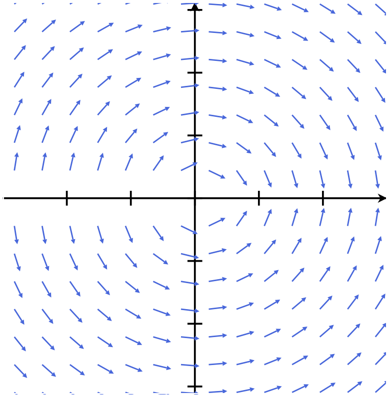

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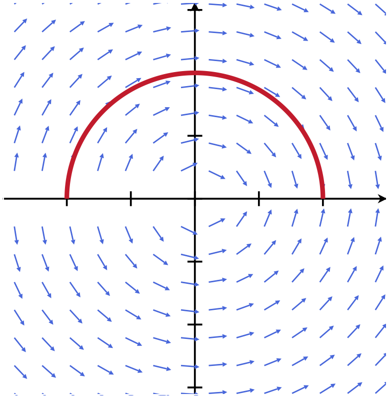

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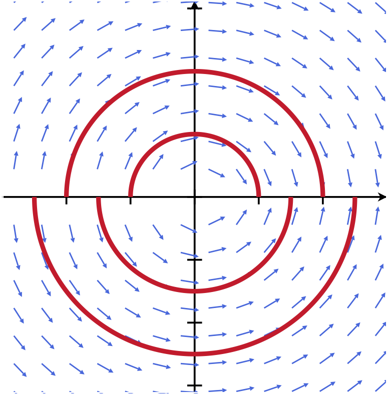

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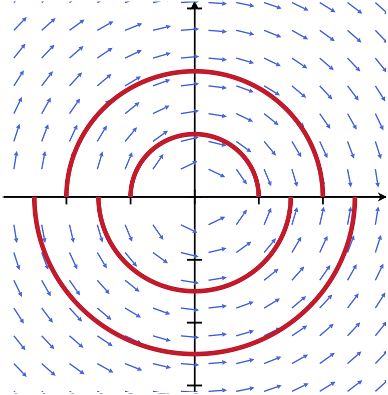

Arrows point in the direction of semicirles! $y = \pm\sqrt{r^2 - x^2}$?

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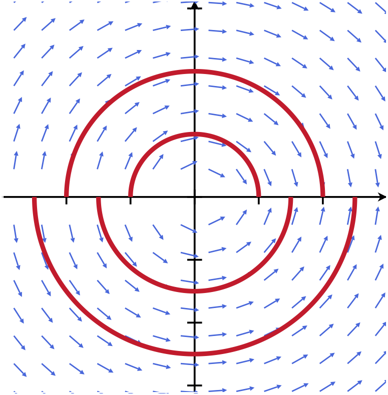

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Check: $$\frac{d}{dx} \pm \sqrt{r^2 - x^2} = \frac{-2x}{\pm 2\sqrt{r^2 - x^2}}$$

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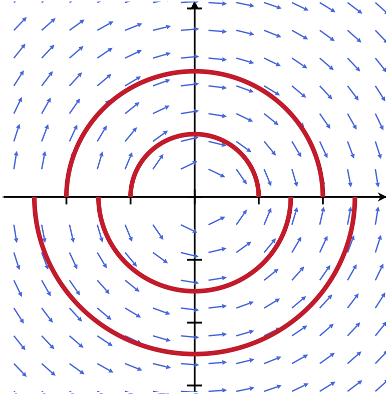

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Check: $$\frac{d}{dx} \pm \sqrt{r^2 - x^2} = \frac{-2x}{\pm 2\sqrt{r^2 - x^2}} = -\frac{x}{y}$$ ☺

Solving explicitly (get a formula!)

We've done. . .

1. Get lucky
   "what's a function you know whose derivative blah blah . . . "
2. Differential equations of the form

$$\frac{dy}{dx} = f(x)$$

   Find the antiderivative!

Today, we'll add

3. Differential equations of the form

$$\frac{dy}{dx} = f(x) * g(y)$$

   Use **"Separation of Variables"**

Separable Equations

A **separable differential equation** is one of the form

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A separable equation is one in which we can put all of the y's and dy's (as products) on one side of the equation and all of the x's and dx's (as products) on the other...

Examples

(1) If $\dfrac{dy}{dx} = -\dfrac{x}{y},$ then $y\ dy = -x\ dx.$

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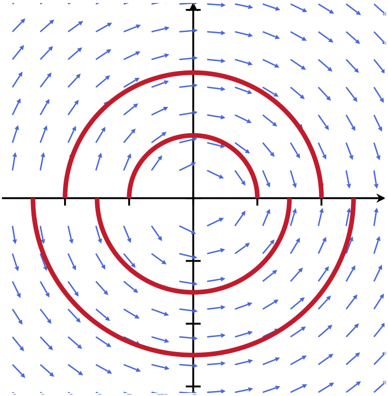
So

$$y = \pm\sqrt{2(-x^2/2 + c_1 - c_2)} = \pm\sqrt{a - x^2}$$

where $a = 2(c_1 - c_2)$.

Examples

Slope field for $\frac{dy}{dx} = -\frac{x}{y}$:



Suggested and checked $y = \pm\sqrt{r^2 - x^2}$

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Find an implicit formula for (2) (with no derivatives left in it)

How many solutions are there?

Existence?

How do I know I even get a solution?
An important result in the theory of differential equations is **Peano's Existence Theorem**, which states...

> If $\frac{dy}{dx} = F(x, y)$ and $y(a) = b$,
> where $F(x, y)$ is continuous in a domain D,
> then there is always **at least one solution** in the domain, and any such solution is differentiable.

Uniqueness?

How do we know that there is not another solution?

> If, additionally, $F(x, y) = f(x)g(y)$, and if g' and f' are continuous, then solution is unique.

Example of non-uniqueness

Suppose $\frac{dy}{dx} = \frac{1}{x}$ and $y(2) = 1$.

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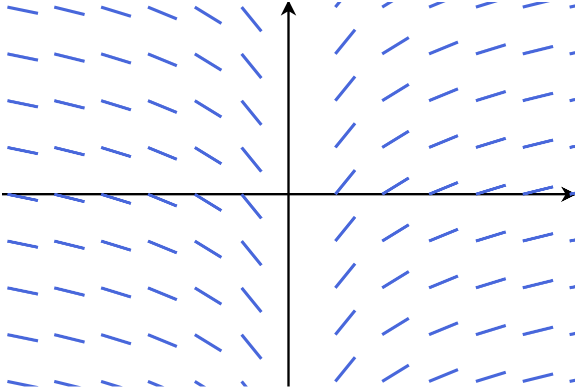

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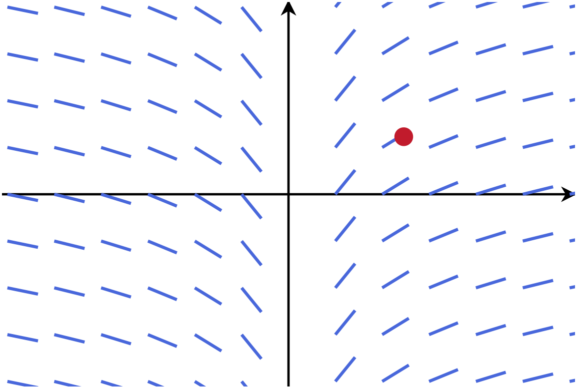

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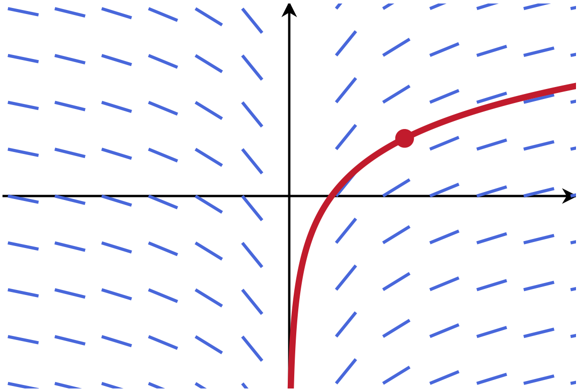

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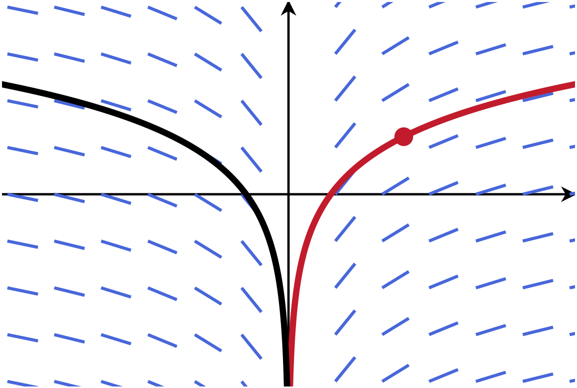

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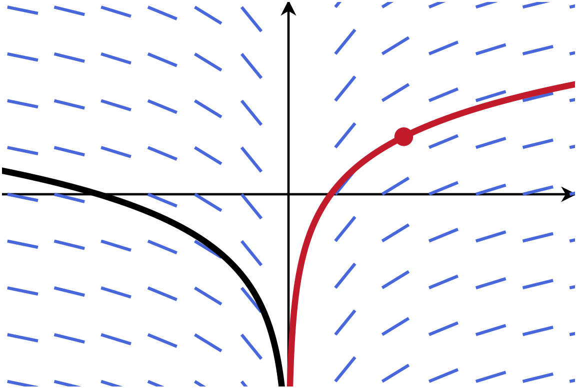

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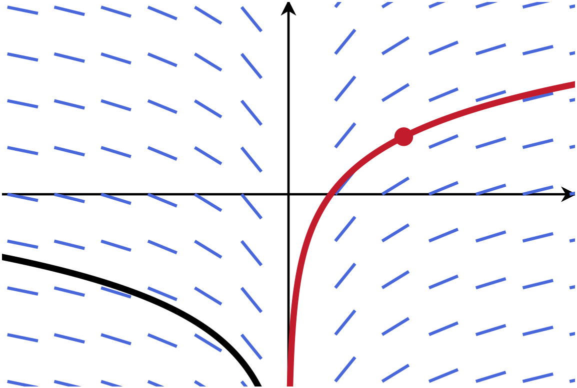

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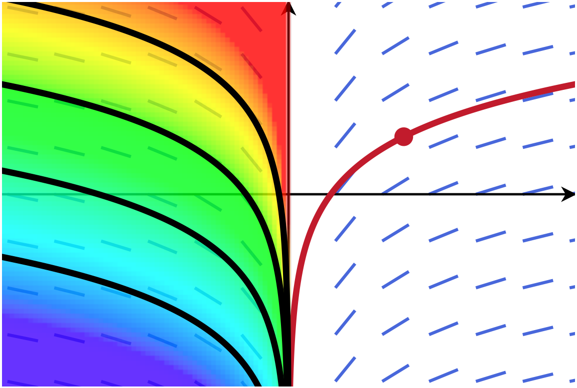
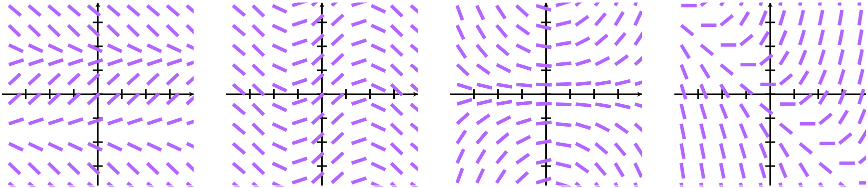


Since $\frac{1}{x}$ is not continuous at $x = 0$, we might have lots of solutions, all that split at 0!

(1) Match the differential equations to the slope fields:

$(A)\ \frac{dy}{dx} = \frac{1}{5}xy \quad (B)\ \frac{dy}{dx} = x+y \quad (C)\ \frac{dy}{dx} = \cos(x) \quad (D)\ \frac{dy}{dx} = \cos(y)$

(a) (b) (c) (d)



(2) Solve the initial value problems

$$(a)\ \frac{dy}{dx} = \frac{1}{5}xy, \qquad y(0) = 2;$$

$$(b)\ \frac{dy}{dx} = \sin(x)/y^2, \qquad y(0) = 3.$$