Arc Length

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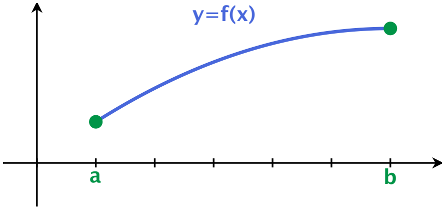

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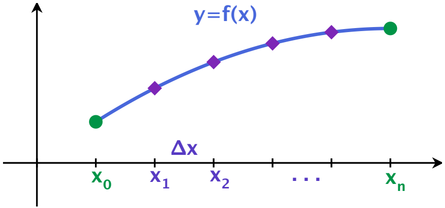

Slice!

$$\ell = \lim_{n\to\infty} \sum_{i=1}^{n} (\text{little length})_i$$

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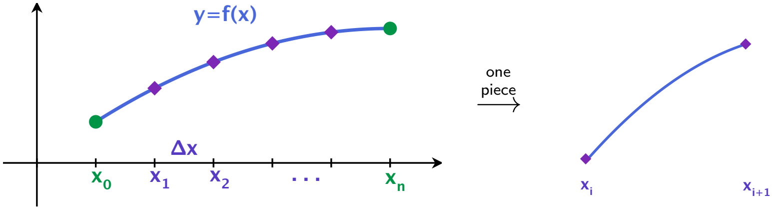

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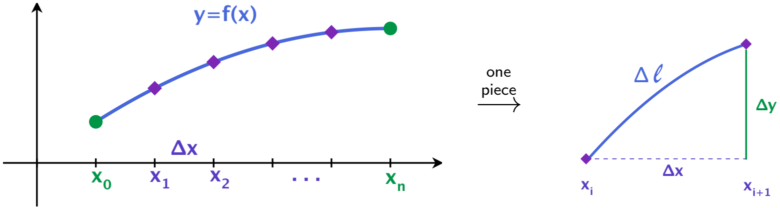

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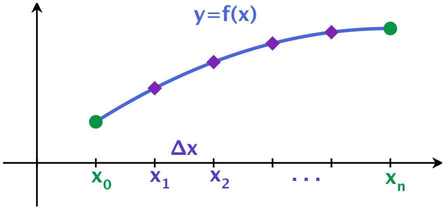
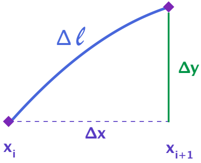


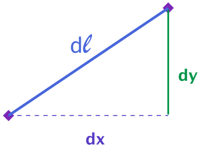
one piece ⟶



Slice!

Let n go to ∞

$\downdownarrows$



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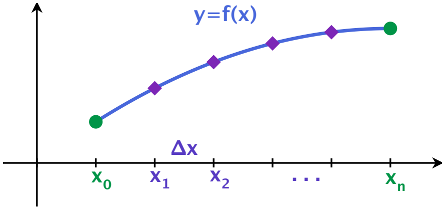

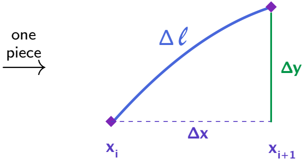

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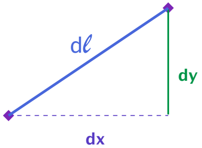
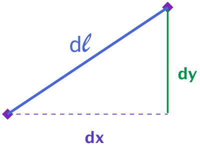


$$\boxed{d\ell = \sqrt{dx^2 + dy^2}}$$

Manipulating into something we can actually calculate...



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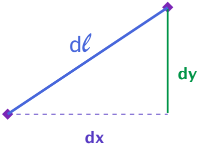

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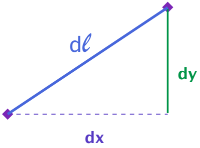

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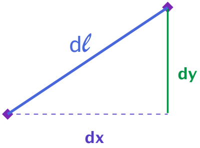

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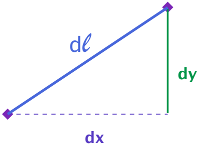

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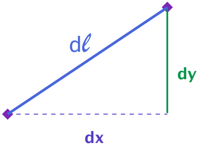

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$$
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&= \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\ dx \\
&= \sqrt{1 + (f'(x))^2}\ dx
\end{aligned}
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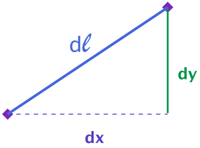

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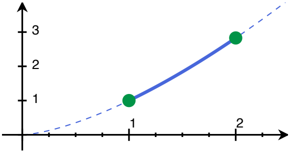
So $$\boxed{\ell = \int_{x=a}^{b} \sqrt{1 + (f'(x))^2}\ dx}$$

Example

Find the length of the arc $y = x^{3/2}$, from $x = 1$ to $x = 2$.



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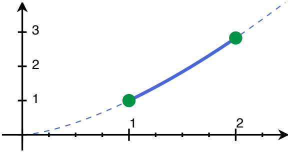

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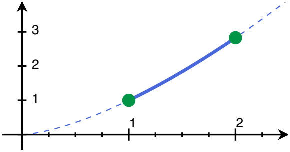

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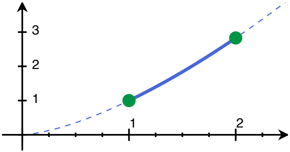

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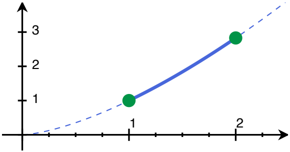

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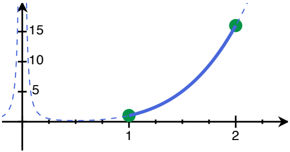
$$= \frac{4}{9}\frac{2}{3}\left(1 + \frac{9}{4}x\right)^{3/2}\Bigg|_{x=1}^{2} = \boxed{\frac{8}{27}\left(\left(1 + \frac{9}{2}\right)^{3/2} - \left(1 + \frac{9}{4}\right)^{3/2}\right)}$$

Find the length of the curve $y = x^4 + \frac{1}{32x^2}$ from $x = 1$ to $x = 2$.



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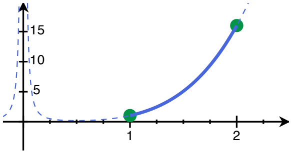

$$f(x) = x^4 + \frac{1}{32}x^{-2} \implies f'(x) = 4x^3 - \frac{1}{16}x^{-3} = \frac{64x^6 - 1}{16x^3}$$

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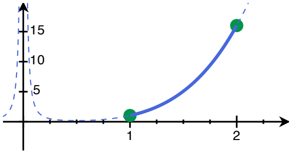

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Let $A = (2x)^3 = 8x^3$ and so $A^2 = 64x^6$, and $f'(x) = \frac{A^2-1}{2A}$.

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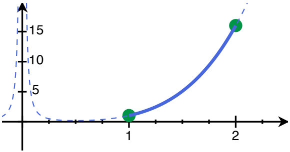

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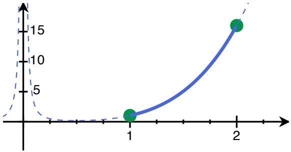

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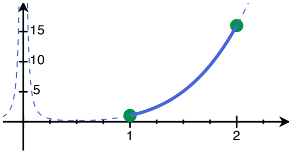

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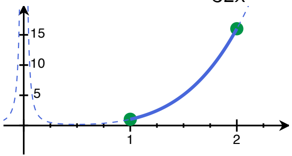

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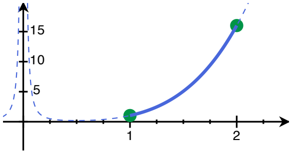

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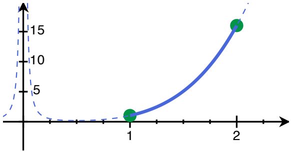

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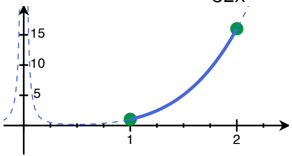

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$$= \frac{A^4+2A^2+1}{4A^2} = \left(\frac{A^2+1}{2A}\right)^2 = \left(\frac{64x^6+1}{16x^3}\right)^2 = (4x^3 + \tfrac{1}{16}x^{-3})^2$$

Most of the time, the resulting integral is “hard” (not elementary)

Set up (but do not integrate) the integrals which compute the length of the following functions:

1. $f(x) = x^2$ from $x = -3$ to 2
2. $f(x) = x^2 + 5$ from $x = -3$ to 2
3. $f(x) = -x^2 + \pi$ from $x = -3$ to 2
4. $f(x) = \sin(x)$ from $x = 0$ to $\frac{\pi}{2}$
5. $f(x) = e^x$ from $x = 0$ to 1
6. $f(x) = \sqrt{1 - x^2}$ from $x = -1$ to 1

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