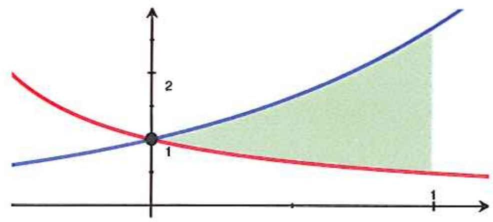
Worksheet: Area between curves

Example 1:
Find the area of the region between $y = e^x$ and $y = 1/(1+x)$ on the interval $[0, 1]$.



1. Check for intersection points (verify algebraically that $x = 0$ is the only intersection by setting $e^x = \frac{1}{x+1}$).
2. Decide which function is on top ($f(x)$) and which function is on bottom ($g(x)$).
3. Calculate $\int_0^1 f(x) - g(x) dx$.

Check: What if you get a negative answer?

1.(a) $x=0$ is an intersection: $e^0 = 1 = \frac{1}{1+0}$ ✓

(b) it's the only one:

on $[0,1]$ e^x is increasing and $\frac{1}{1+x}$ is decreasing, so they can intersect at most once!

2. plug in $\frac{1}{2}$: $e^{1/2} > \frac{1}{1+1/2}$. so $e^x > \frac{1}{1+x}$.

3. $A = \int_0^1 e^x - \frac{1}{1+x} \, dx$.

$= e^x - \ln|1+x| \Big|_{x=0}^{1}$

$= e - \ln(2) - (e^0 - \underbrace{\ln(1)}_{0})$

$= e - \ln(2) - 1$

$\int e^x dx = e^x$

$\int \frac{1}{1+x} dx$: let $u = 1+x$, so $du = dx$

$\hookrightarrow = \int \frac{1}{u} du = \ln|u| + c$

$= \ln|1+x| + c$

Example 2:
Find the area of the region bounded by $y = x^2 - 2x$ and $y = 4 - x^2$.

1. Check for intersection points (Solve $x^2 - 2x = 4 - x^2$). There will be two, a and b; this is where the functions cross.
2. Between this two points, which function is on top ($f(x)$) and which function is on bottom ($g(x)$).
3. Calculate $\int_a^b f(x) - g(x)dx$.

Check: What if you get a negative answer?

1. $x^2 - 2x = 4 - x^2$ so $2x^2 - 2x - 4 = 0$

$2(x+1)(x-2) =$

$x = -1, 2$

2. plug in 0:

$\underbrace{x^2 - 2x \Big|_{x=0}}_{\text{bot}} = 0 < \underbrace{4 - x^2 \Big|_{x=0}}_{\text{top}} = 4$

3. $\int_{-1}^{2} 4 - x^2 - (x^2 - 2x)\,dx = \int_{-1}^{2} 4 - 2x^2 + 2x\,dx$

$= 4x - \frac{2}{3}x^3 + x^2 \Big|_{-1}^{2}$

$= 8 - \frac{2}{3}\cdot 8 + 4 - \left(-4 + \frac{2}{3} + 1\right)$

$= 12 - \frac{2}{3}\cdot 9 + 3$

$= \boxed{9}$

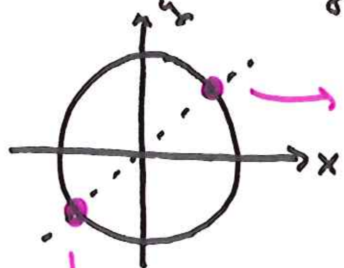
Example 3
Find the area between $\sin x$ and $\cos x$ on $[-3\pi/4, 5\pi/4]$.

(Hint: There are several places where $\sin(x) = \cos(x)$. For example, $x = \pi/4$.)

Where do they intersect?

go back to the unit circle:

Where are the x & y coordinates equal?



$\frac{\pi}{4}, \frac{\pi}{4} \pm 2\pi, \frac{\pi}{4} \pm 4\pi, \ldots$

$\pi + \frac{\pi}{4}, \pi + \frac{\pi}{4} \pm 2\pi, \pi + \frac{\pi}{4} \pm 4\pi \ldots.$

in the interval $\left[-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$, that's

$x = -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$. (see graph w/ answers)

Check some intermediate points:

$\underline{x = 0}$; $x = \pi$

$1 = \cos(0) > \sin(0) = 0$

top

$-1 = \cos(\pi) < \sin(0) = 1$

top

$$A = \int_{-3\pi/4}^{\pi/4} \cos(x) - \sin(x)\,dx + \int_{\pi/4}^{5\pi/4} \sin(x) - \cos(x)$$

$$= \sin(x) + \cos(x)\Big|_{x=-3\pi/4}^{\pi/4} + \Big(-\cos(x) - \sin(x)\Big)\Big|_{x=\pi/4}^{5\pi/4}$$

$$= \frac{1}{\sqrt{2}}\big(1 + 1 - (-1-1)\big) + \frac{1}{\sqrt{2}}\big(-(-1) - (-1) - (-1-1)\big) = \frac{8}{\sqrt{2}} = \boxed{4\sqrt{2}}$$

Example 4
Calculate the area under the curve $y = \arccos(x)$ from $x = 0$ to $x = 1$.

Hint: Since we don't know $\int \arccos(x)\, dx$, use the fact that $y = \arccos(x)$ if and only if $\cos(y) = x$.
(1) Draw graphs of both $y = \arccos(x)$ and $x = \cos(y)$ on separate axes (the first with x on the horizontal axis, and the second with y on the horizontal axis).
(2) What integral, involving $\cos(y)$ (and endpoints for y's instead of x's, and with a dy instead of a dx) will compute the same area as $\int_0^1 \arccos(x)\, dx$?

$$y = \arccos(x) \longleftrightarrow x = \cos(y)$$

(see picture w/ answers)

$$A = \int_{y=0}^{\pi/2} f_{right} - f_{left}\, dy$$

$$= \int_{y=0}^{\pi/2} \cos(y) - 0\, dy = \sin(y)\Big|_{y=0}^{\pi/2}$$

$$= \sin(\pi/2) - \sin(0)$$

$$= \boxed{1}$$

Example 5
Calculate the area under the curve $y = \arcsin(x)$ from $x = 0$ to $x = 1$.

Hint: Similar to Example 4, but be careful! Be sure to draw the pictures before writing down the corresponding integrals!

$$y = \sin^{-1}(x) \longleftrightarrow x = \sin(y)$$

(see picture w/ answers)

$$A = \int_{y=0}^{\pi/2} f_{right} - f_{left}\, dy$$

$$= \int_{y=0}^{\pi/2} 1 - \sin(y)\, dy$$

$$= y + \cos(y) \Big|_{y=0}^{\pi/2}$$

$$= \pi/2 + \underbrace{\cos(\pi/2)}_{0} - (0 + \underbrace{\cos(0)}_{1})$$

$$= \boxed{\pi/2 - 1}$$