The FUNDAMENTAL Theorem of Calculus (yay!)

Warm-up

Suppose a particle is traveling at velocity $v(t) = t^2$ from $t = 1$ to $t = 2$. if the particle starts at $y(0) = y_0$,

1. what is the function $y(t)$ which gives the particles position as a function of time (will have a y_0 in it)?
2. how far does the particle travel from $t = 1$ to $t = 2$?

Compare your answer to the upper and lower estimates of the area under the curve $f(x) = x^2$ from $x = 1$ to $x = 2$:

Upper

$$\sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 * \left(\frac{1}{n}\right)$$

Lower

$$\sum_{i=0}^{n-1} \left(1 + \frac{i}{n}\right)^2 * \left(\frac{1}{n}\right)$$

n	Upper	Lower
10	2.485	2.185
100	2.34835	2.31835
1000	2.33483	2.33183

The Fundamental Theorem of Calculus

Theorem (the baby case)

If $F(x)$ is any function satisfying $\frac{d}{dx}F(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Q. What is $\int_1^2 x^2 dx$?

A. $F(x) = \frac{x^3}{3} + C$
So

$$\int_1^2 x^2 dx = F(2) - F(1) = \left(\frac{2^3}{3} + C\right) - \left(\frac{1^3}{3} + C\right)$$
$$= \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3} \approx 2.333}$$

The Fundamental Theorem of Calculus

Theorem (the baby case)

If $F(x)$ is any function satisfying $\frac{d}{dx}F(x) = f(x)$, then

$$\int_a^b f(x)dx = F(x)\Big|_{x=a}^{b} = F(b) - F(a)$$

Q. What is $\int_1^2 x^2 dx$?

A. $F(x) = \frac{x^3}{3}$
So

$$\int_1^2 x^2 dx = \frac{x^3}{3}\Big|_{x=1}^{2} = \left(\frac{2^3}{3}\right) - \left(\frac{1^3}{3}\right)$$
$$= \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3} \approx 2.333} \qquad \text{(same answer!)}$$

Examples

Use the fundamental theorem of calculus,

$$\int_a^b f(x)dx = F(b) - F(a)$$

to calculate

1. $\int_2^3 3x \; dx$
2. $\int_{-1}^1 x^3 \; dx$
3. $\int_0^\pi \sin(x) \; dx$
4. $\int_\pi^0 \sin(x) \; dx$

The Fundamental Theorem of Calculus

Theorem (the big case)

If $F(x)$ is any function satisfying $\frac{d}{dt}F(t) = f(t)$, then

$$\int_{a(x)}^{b(x)} f(t)dt = F(t)\Big|_{t=a(x)}^{b(x)} = F(b(x)) - F(a(x))$$

Q. What is $\int_{\sin(x)}^{\ln(x)} t^2 dt$?

A. $F(t) = \frac{1}{3}t^3$.

So

$$\int_{\sin(x)}^{\ln(x)} t^2 dt = \frac{1}{3}t^3\Bigg|_{t=\sin(x)}^{\ln(x)} = \left(\frac{1}{3}(\ln(x))^3\right) - \left(\frac{1}{3}(\sin(x))^3\right).$$

Examples

Use the fundamental theorem of calculus,

$$\int_{a(x)}^{b(x)} f(t)dt = F(b(x)) - F(a(x))$$

to calculate

1. $\int_{\sin(x)}^{\cos(x)} 3t \ dt$
2. $\int_{x+1}^{5x^2-3} t^3 \ dt$
3. $\int_{\arccos(x)}^{0} \sin(t) \ dt$

For reference, we calculated $\int_{a(x)}^{b(x)} f(t)\,dt$ where

$$f(t) = t^2 \qquad a(x) = \sin(x) \qquad b(x) = \ln(x).$$

Notice:

$$\frac{d}{dx}\left(\frac{1}{3}(\ln(x))^3 - \frac{1}{3}(\sin(x))^3\right) = \frac{1}{x}(\ln(x))^2 - \cos(x)(\sin(x))^2$$
$$= b'(x)f(b(x)) - a'(x)f(a(x)).$$

In general:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = b'(x)f(b(x)) - a'(x)f(a(x)).$$

(Don't even have to know $F(t)$!)

Why?

Example: Calculate $\frac{d}{dx}\int_{\tan(x)}^{\sin(x)} e^{t^2}\,dt$.

Answer: We can't even calculate $\int e^{t^2}\,dt$!

(There is no elementary function $F(t)$ which satisfies $F'(t) = e^{t^2}$)

But we know $\int e^{t^2}\,dt$ is a function. Call it $F(t)$.

So $\int_{\tan(x)}^{\sin(x)} e^{t^2}\,dt = F(\sin(x)) - F(\tan(x))$.

Therefore
$$\frac{d}{dx}\int_{\tan(x)}^{\sin(x)} e^{t^2}\,dt = \frac{d}{dx}\left(F(\sin(x)) - F(\tan(x))\right)$$
$$= \cos(x)F'(\sin(x)) - \sec^2(x)F'(\tan(x))$$
$$= \cos(x)f(\sin(x)) - \sec^2(x)f(\tan(x))$$
$$= \cos(x)e^{(\sin(x))^2} - \sec^2(x)e^{(\tan(x))^2}$$

Part 2: Integration by substitution

Warmup

Fill in the blank:

1. Since $\dfrac{d}{dx}\cos(x^2+1) = $ ______________,

   so $\displaystyle\int$ ______________ $dx = \cos(x^2+1) + C$.

2. Since $\dfrac{d}{dx}\ln|\cos(x)| = $ ________,

   so $\displaystyle\int$ ________ $dx = \ln|\cos(x)| + C$.

(Example: $\frac{d}{dx}x^3 dx = 3x^2$, so $\int 3x^2 dx = x^3 + C$.)

Undoing chain rule

In general:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) * g'(x),$$

so $$\int f'(g(x)) * g'(x) dx = f(g(x)) + C.$$

Example: Calculate the (extremely suggestively written) integral

$$\int \cos(x^3 + 5x - 10) * (3x^2 + 5 * 1 + 0) dx = \sin(x^3 + 5x - 10) + C$$

Check: $\frac{d}{dx} \sin(x^3 + 5x - 10) = \cos(x^3 + 5x - 10) * (3x^2 + 5)$✓

Less obvious chain rules.

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx:

Examples:

$$\int \frac{\cos(x)}{\sin(x) + 1} dx = \int \frac{1}{\sin(x) + 1} * \cos(x)\ dx$$

$$\int x\sqrt{x^2 + 1} dx = \int \frac{1}{2} \sqrt{x^2 + 1} * 2x\ dx$$

$$\int \frac{\cos(\sqrt{x})}{2\sqrt{x}} dx = \int \cos(\sqrt{x}) * \frac{1}{2\sqrt{x}}\ dx$$

Method of Substitution

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

Example: $$\int \frac{\cos(x)}{\sin(x)+1}dx$$

Let $u = g(x)$.	Let $u = \sin(x) + 1$
Calculate du.	$\frac{du}{dx} = \cos(x)$ so $du = \cos(x)dx$
Clear out all of the x's, replacing them with u's.	$\int \frac{1}{u}du$
Calculate the new integral.	$\int \frac{1}{u}du = \ln\lvert u\rvert + C$
Substitute back into x's.	$\ln\lvert u\rvert + C = \ln\lvert\sin(x)+1\rvert + C$

Check $\frac{d}{dx}\ln\lvert\sin(x)+1\rvert + C = \frac{1}{\sin(x)+1} * \cos(x)\checkmark$

Method of Substitution

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

Example: $$\int x\sqrt{x^2+1}\ dx$$

Let $u = g(x)$.	Let $u = x^2 + 1$
Calculate $c * du$.	$\frac{du}{dx} = 2x$ so $\frac{1}{2}du = x\ dx$
Clear out all of the x's, replacing them with u's.	$\int \sqrt{u} * \frac{1}{2}du$
Calculate the new integral.	$\frac{1}{2}\int u^{1/2}du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C$
Substitute back into x's.	$= \frac{1}{3}(x^2+1)^{3/2} + C$

Check $\frac{d}{dx}\frac{1}{3}(x^2+1)^{3/2} + C = \frac{1}{3}\frac{3}{2}(x^2+1)^{1/2} * 2x\checkmark$

Give it a try:

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

$$\text{Example:} \quad \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

Let $u = g(x)$.

Calculate $c * du$.

Clear out all of the x's,
replacing them with u's.

Calculate the new integral.

Substitute back into x's.

Extra practice: Integrals with substitution

(Check answers by taking a derivative and/or on Wolfram Alpha)

1. $\int (2x+9)^5 \, dx$

2. $\int (7-3x)^4 \, dx$

3. $\int \sqrt{3x-5} \, dx$

4. $\int \frac{1}{\sqrt{4x+3}} \, dx$

5. $\int \frac{1}{\sqrt{3-4x}} \, dx$

6. $\int \frac{1}{(2x-3)^{3/2}} \, dx$

7. $\int \frac{4x}{2x^2+3} \, dx$

8. $\int \frac{x+1}{x^2+2x-3} \, dx$

9. $\int \frac{4x-5}{2x^2-5x+1} \, dx$

10. $\int \frac{9x^2-4x+5}{3x^3-2x^2+5x+1} \, dx$

11. $\int \frac{2x+3}{\sqrt{x^2+3x-2}} \, dx$

12. $\int \frac{2x-1}{\sqrt{x^2-x-1}} \, dx$

13. $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

14. $\int \frac{dx}{\sqrt{1-3x}-\sqrt{5-3x}}$

15. $\int \frac{x^2}{1+x^6} \, dx$

16. $\int \frac{x^3}{1+x^8} \, dx$

17. $\int \frac{x}{1+x^4} \, dx$

18. $\int \frac{x^5}{\sqrt{1+x^3}} \, dx$

19. $\int \frac{x}{\sqrt{1+x}} \, dx$

20. $\int \frac{1}{x\sqrt{x^4-1}} \, dx$

21. $\int x\sqrt{x-1} \, dx$

22. $\int (1-x)\sqrt{1+x} \, dx$

23. $\int x\sqrt{x^2-1} \, dx$

24. $\int x\sqrt{3x-2} \, dx$

25. $\int (2x-3)\sqrt{x^2-3x+5} \, dx$

26. $\int \frac{dx}{3-5x}$

27. $\int \sqrt{1+x} \, dx$

28. $\int \sin 3x \, dx$

29. $\int \cos(5+6x) \, dx$

30. $\int \sin(5-3x) \, dx$

31. $\int \csc^2(2x+5)\ dx$

32. $\int \sin x \cos x\ dx$

33. $\int \sin^3 x \cos x\ dx$

34. $\int \sqrt{\cos x} \sin x\ dx$

35. $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\ dx$

36. $\int \sin(ax+b)\cos(ax+b)\ dx$

37. $\int \cos^3 x\ dx$

38. $\int (1/x^2)\cos(1/x)\ dx$

39. $\int 2x \sin(x^2+1)\ dx$

40. $\int \frac{\tan \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}\ dx$

41. $\int \frac{\sec^2 x}{1+\tan x}\ dx$

42. $\int \frac{\sin x}{1+\cos x}\ dx$

43. $\int \frac{\sin x}{2+3\cos x}\ dx$

44. $\int \frac{\sin 2x}{a^2+b^2\sin^2 x}\ dx$

45. $\int \frac{\sin 2x}{a^2\cos^2 x+b^2\sin^2 x}\ dx$

46. $\int \frac{2\cos x-3\sin x}{3\cos x+2\sin x}\ dx$

47. $\int \frac{1+\cos x}{(x+\sin x)^3}\ dx$

48. $\int \frac{\sin x}{(1+\cos x)^2}\ dx$

49. $\int x^2 \sin x^3\ dx$

50. $\int \frac{\sin x}{\sin x-\cos x}\ dx$

51. $\int \frac{dx}{1-\tan x}$

52. $\int \frac{dx}{1-\cot x}$

53. $\int \frac{\cos 2x}{(\sin x+\cos x)^2}$

54. $\int \frac{\cos x-\sin x}{(1+\sin 2x)}$

55. $\int x \sin^3(x^2)\cos(x^2)\ dx$

56. $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\ dx$

57. $\int 2x \sec^3(x^2+3)\tan(x^2+3)\ dx$

58. $\int \frac{\sin 2x}{(a+b\cos 2x)^2}\ dx$