

The FUNDAMENTAL Theorem of Calculus (yay!)

Warm-up

Suppose a particle is traveling at velocity $v(t) = t^2$ from $t = 1$ to $t = 2$. if the particle starts at $y(0) = y_0$,

1. what is the function $y(t)$ which gives the particles position as a function of time (will have a y_0 in it)?
2. how far does the particle travel from $t = 1$ to $t = 2$?

Compare your answer to the upper and lower estimates of the area under the curve $f(x) = x^2$ from $x = 1$ to $x = 2$:

$$\begin{array}{cc} \text{Upper} & \text{Lower} \\ \sum_{i=1}^n \left(1 + \frac{i}{n}\right)^2 * \left(\frac{1}{n}\right) & \sum_{i=0}^{n-1} \left(1 + \frac{i}{n}\right)^2 * \left(\frac{1}{n}\right) \end{array}$$

n	Upper	Lower
10	2.485	2.185
100	2.34835	2.31835
1000	2.33483	2.33183

The Fundamental Theorem of Calculus

Theorem (the baby case)

If $F(x)$ is any function satisfying $\frac{d}{dx}F(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Q. What is $\int_1^2 x^2 dx$?

A. $F(x) = \frac{x^3}{3} + C$

So

$$\begin{aligned}\int_1^2 x^2 dx &= F(2) - F(1) = \left(\frac{2^3}{3} + C\right) - \left(\frac{1^3}{3} + C\right) \\ &= \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3} \approx 2.333}\end{aligned}$$

The Fundamental Theorem of Calculus

Theorem (the baby case)

If $F(x)$ is any function satisfying $\frac{d}{dx}F(x) = f(x)$, then

$$\int_a^b f(x)dx = F(x) \Big|_{x=a}^b = F(b) - F(a)$$

Q. What is $\int_1^2 x^2 dx$?

A. $F(x) = \frac{x^3}{3}$

So

$$\begin{aligned} \int_1^2 x^2 dx &= \frac{x^3}{3} \Big|_{x=1}^2 = \left(\frac{2^3}{3}\right) - \left(\frac{1^3}{3}\right) \\ &= \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3} \approx 2.333} \quad (\text{same answer!}) \end{aligned}$$

Examples

Use the fundamental theorem of calculus,

$$\int_a^b f(x) dx = F(b) - F(a)$$

to calculate

$$1. \int_2^3 3x \, dx = 3 \frac{x^2}{2} \Big|_{x=2}^3 = 3 * \frac{9}{2} - 3 * \frac{4}{2} = 15/2$$

$$2. \int_{-1}^1 x^3 \, dx = \frac{x^4}{4} \Big|_{x=-1}^1 = \frac{1^4}{4} - \frac{(-1)^4}{4} = 0 \quad (\text{odd function!!})$$

$$3. \int_0^\pi \sin(x) \, dx = -\cos(x) \Big|_{x=0}^\pi = -\cos(\pi) - (-\cos(0)) \\ = -(-1) - (-1) = 2$$

$$4. \int_\pi^0 \sin(x) \, dx = -\cos(x) \Big|_{x=\pi}^0 = -\cos(0) - (-\cos(\pi)) \\ = -(1) - (-(-1)) = -2$$

The Fundamental Theorem of Calculus

Theorem (the big case)

If $F(x)$ is any function satisfying $\frac{d}{dt}F(t) = f(t)$, then

$$\int_{a(x)}^{b(x)} f(t)dt = F(t) \Big|_{t=a(x)}^{b(x)} = F(b(x)) - F(a(x))$$

Q. What is $\int_{\sin(x)}^{\ln(x)} t^2 dt$?

A. $F(t) = \frac{1}{3}t^3$.

So

$$\int_{\sin(x)}^{\ln(x)} t^2 dt = \frac{1}{3}t^3 \Big|_{t=\sin(x)}^{\ln(x)} = \left(\frac{1}{3}(\ln(x))^3\right) - \left(\frac{1}{3}(\sin(x))^3\right).$$

Examples

Use the fundamental theorem of calculus,

$$\int_{a(x)}^{b(x)} f(t) dt = F(b(x)) - F(a(x))$$

to calculate

$$1. \int_{\sin(x)}^{\cos(x)} 3t \, dt = \frac{3}{2} t^2 \Big|_{t=\sin(x)}^{\cos(x)} = \frac{3}{2} (\cos(x))^2 - \frac{3}{2} (\sin(x))^2$$

$$2. \int_{x+1}^{5x^2-3} t^3 \, dt = \frac{1}{4} t^4 \Big|_{t=x+1}^{5x^2-3} = \frac{1}{4} (5x^2 - 3)^4 - \frac{1}{4} (x + 1)^4$$

$$3. \int_{\arccos(x)}^0 \sin(t) \, dt = -\cos(t) \Big|_{t=\arccos(x)}^0 \\ = -\cos(0) - (-\cos(\arccos(x))) = -(1) - (-(x)) = x - 1$$

For reference, we calculated $\int_{a(x)}^{b(x)} f(t) dt$ where

$$f(t) = t^2 \quad a(x) = \sin(x) \quad b(x) = \ln(x).$$

Notice:

$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{3}(\ln(x))^3 - \frac{1}{3}(\sin(x))^3 \right) &= \frac{1}{x}(\ln(x))^2 - \cos(x)(\sin(x))^2 \\ &= b'(x)f(b(x)) - a'(x)f(a(x)). \end{aligned}$$

In general:

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) dt = b'(x)f(b(x)) - a'(x)f(a(x)).$$

(Don't even have to know $F(t)$!)

Why?

Example: Calculate $\frac{d}{dx} \int_{\tan(x)}^{\sin(x)} e^{t^2} dt$.

Answer: We can't even calculate $\int e^{t^2} dt$!

(There is no elementary function $F(t)$ which satisfies $F'(t) = e^{t^2}$)

But we know $\int e^{t^2} dt$ is a function. Call it $F(t)$.

So $\int_{\tan(x)}^{\sin(x)} e^{t^2} dt = F(\sin(x)) - F(\tan(x))$.

$$\begin{aligned} \text{Therefore } \frac{d}{dx} \int_{\tan(x)}^{\sin(x)} e^{t^2} dt &= \frac{d}{dx} (F(\sin(x)) - F(\tan(x))) \\ &= \cos(x)F'(\sin(x)) - \sec^2(x)F'(\tan(x)) \\ &= \cos(x)f(\sin(x)) - \sec^2(x)f(\tan(x)) \\ &= \cos(x)e^{(\sin(x))^2} - \sec^2(x)e^{(\tan(x))^2} \end{aligned}$$

Part 2: Integration by substitution

Warmup

Fill in the blank:

1. Since $\frac{d}{dx} \cos(x^2 + 1) = \underline{-2x \sin(x^2 + 1)}$,

$$\text{so } \int \underline{-2x \sin(x^2 + 1)} dx = \cos(x^2 + 1) + C.$$

2. Since $\frac{d}{dx} \ln |\cos(x)| = \underline{-\frac{\sin(x)}{\cos(x)}}$,

$$\text{so } \int \underline{-\frac{\sin(x)}{\cos(x)}} dx = \ln |\cos(x)| + C.$$

(Example: $\frac{d}{dx} x^3 dx = 3x^2$, so $\int 3x^2 dx = x^3 + C$.)

Undoing chain rule

In general:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) * g'(x),$$

so
$$\int f'(g(x)) * g'(x) dx = f(g(x)) + C.$$

Example: Calculate the (extremely suggestively written) integral

$$\int \cos(x^3 + 5x - 10) * (3x^2 + 5 * 1 + 0) dx = \sin(x^3 + 5x - 10) + C$$

Check:
$$\frac{d}{dx} \sin(x^3 + 5x - 10) = \cos(x^3 + 5x - 10) * (3x^2 + 5) \checkmark$$

Less obvious chain rules.

Look for a buried function $g(x)$ and its derivative $g'(x)$ which can be paired with dx :

Examples:

$$\int \frac{\cos(x)}{\sin(x) + 1} dx = \int \frac{1}{\sin(x) + 1} * \cos(x) dx$$

$$\int x\sqrt{x^2 + 1} dx = \int \frac{1}{2}\sqrt{x^2 + 1} * 2x dx$$

$$\int \frac{\cos(\sqrt{x})}{2\sqrt{x}} dx = \int \cos(\sqrt{x}) * \frac{1}{2\sqrt{x}} dx$$

Method of Substitution

Look for a buried function $g(x)$ and its derivative $g'(x)$ which can be paired with dx .

Example:
$$\int \frac{\cos(x)}{\sin(x) + 1} dx$$

Let $u = g(x)$.

Let $u = \sin(x) + 1$

Calculate du .

$$\frac{du}{dx} = \cos(x) \text{ so } du = \cos(x) dx$$

Clear out all of the x 's, replacing them with u 's.

$$\int \frac{1}{u} du$$

Calculate the new integral.

$$\int \frac{1}{u} du = \ln |u| + C$$

Substitute back into x 's.

$$\ln |u| + C = \ln |\sin(x) + 1| + C$$

Check $\frac{d}{dx} \ln |\sin(x) + 1| + C = \frac{1}{\sin(x)+1} * \cos(x) \checkmark$

Method of Substitution

Look for a buried function $g(x)$ and its derivative $g'(x)$ which can be paired with dx .

Example: $\int x\sqrt{x^2 + 1} dx$

Let $u = g(x)$.

Let $u = x^2 + 1$

Calculate $c * du$.

$$\frac{du}{dx} = 2x \text{ so } \frac{1}{2} du = x dx$$

Clear out all of the x 's, replacing them with u 's.

$$\int \sqrt{u} * \frac{1}{2} du$$

Calculate the new integral.

$$\frac{1}{2} \int u^{1/2} du = \frac{1}{2} \left(\frac{2}{3} u^{3/2} \right) + C$$

Substitute back into x 's.

$$= \frac{1}{3} (x^2 + 1)^{3/2} + C$$

Check $\frac{d}{dx} \frac{1}{3} (x^2 + 1)^{3/2} + C = \frac{1}{3} \frac{3}{2} (x^2 + 1)^{1/2} * 2x \checkmark$
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Give it a try:

Look for a buried function $g(x)$ and its derivative $g'(x)$ which can be paired with dx .

Example:
$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

Let $u = g(x)$.

Calculate $c * du$.

Clear out all of the x 's, replacing them with u 's.

Calculate the new integral.

Substitute back into x 's.

Give it a try:

Look for a buried function $g(x)$ and its derivative $g'(x)$ which can be paired with dx .

Example: $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Let $u = g(x)$.

Let $u = \sqrt{x}$

Calculate $c * du$.

Clear out all of the x 's,
replacing them with u 's.

Calculate the new integral.

Substitute back into x 's.

Give it a try:

Look for a buried function $g(x)$ and its derivative $g'(x)$ which can be paired with dx .

Example: $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Let $u = g(x)$.

Let $u = \sqrt{x}$

Calculate $c * du$.

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \quad \text{so} \quad 2du = \frac{1}{\sqrt{x}} dx$$

Clear out all of the x 's, replacing them with u 's.

Calculate the new integral.

Substitute back into x 's.

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Clear out all of the x 's, replacing them with u 's.

$$\int e^u * 2du$$

Calculate the new integral.

Substitute back into x 's.

Give it a try:

Look for a buried function $g(x)$ and its derivative $g'(x)$ which can be paired with dx .

Example: $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

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Clear out all of the x 's, replacing them with u 's.

$$\int e^u * 2du$$

Calculate the new integral.

$$2 \int e^u du = 2e^u + C$$

Substitute back into x 's.

Give it a try:

Look for a buried function $g(x)$ and its derivative $g'(x)$ which can be paired with dx .

Example: $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Let $u = g(x)$.

Let $u = \sqrt{x}$

Calculate $c * du$.

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \quad \text{so} \quad 2du = \frac{1}{\sqrt{x}} dx$$

Clear out all of the x 's, replacing them with u 's.

$$\int e^u * 2du$$

Calculate the new integral.

$$2 \int e^u du = 2e^u + C$$

Substitute back into x 's.

$$= 2e^{\sqrt{x}} + C$$

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Look for a buried function $g(x)$ and its derivative $g'(x)$ which can be paired with dx .

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Clear out all of the x 's, replacing them with u 's.

$$\int e^u * 2du$$

Calculate the new integral.

$$2 \int e^u du = 2e^u + C$$

Substitute back into x 's.

$$= 2e^{\sqrt{x}} + C$$

Check $\frac{d}{dx} 2e^{\sqrt{x}} + C = 2e^{\sqrt{x}} * \frac{1}{2} \frac{1}{\sqrt{x}} \checkmark$