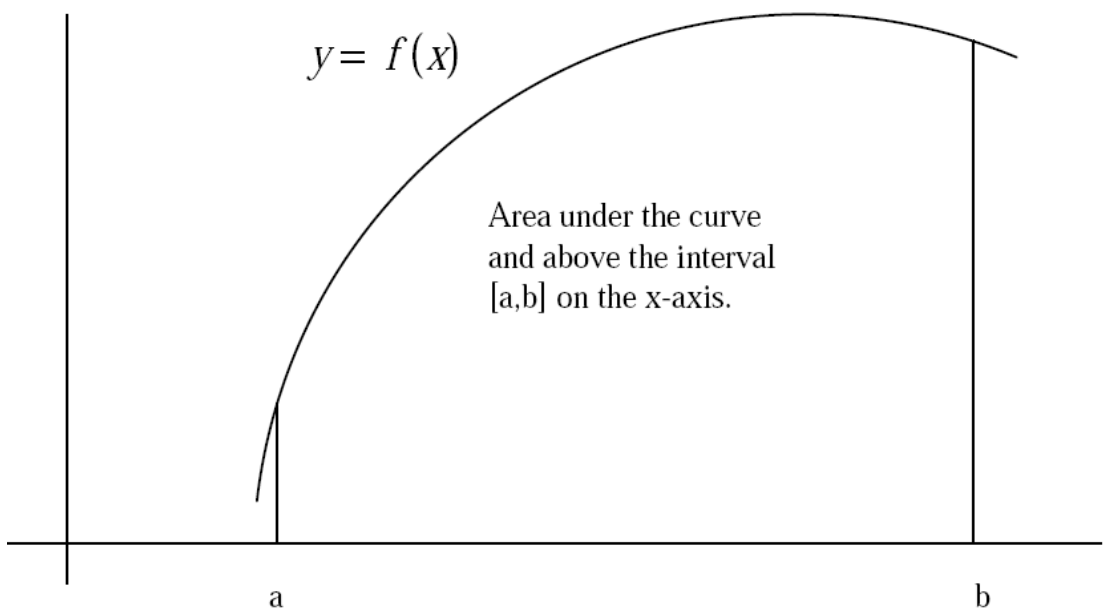
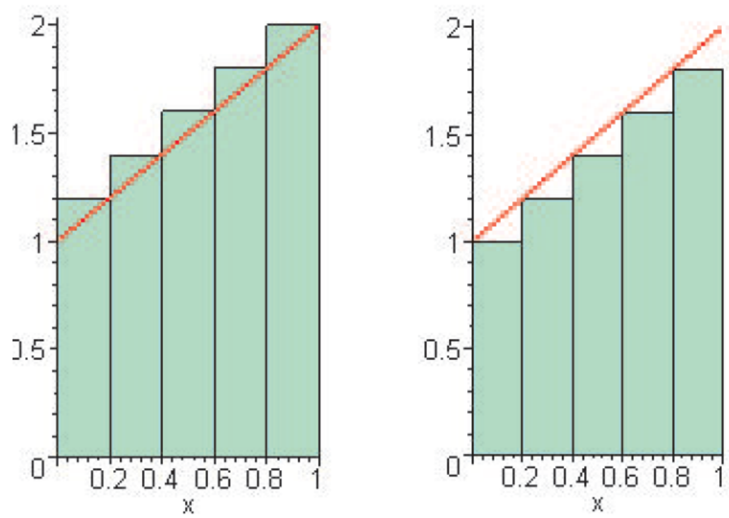
The Definite Integral

The Area Problem



Upper and Lower Sums

Suppose we want to use rectangles to approximate the area under the graph of $y = x + 1$ on the interval $[0, 1]$.



Upper Riemann Sum *Lower Riemann Sum*

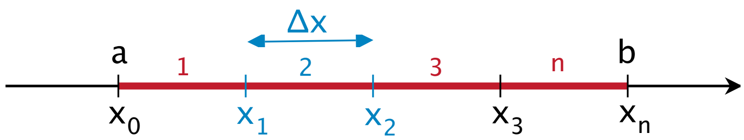
$$31/20 \quad > \quad 1.5 \quad > \quad 29/20$$

As you take more and more smaller and smaller rectangles, if f is nice, both of these will approach the real area.

n	U	L
100	1.505000000	1.495000000
150	1.503333333	1.496666667
200	1.502500000	1.497500000
300	1.501666667	1.498333333
500	1.501000000	1.499000000

In general: finding the Area Under a Curve

Let $y = f(x)$ be given and defined on an interval $[a, b]$.



Break the interval into n equal pieces.

Label the endpoints of those pieces $x_0, x_1, \ldots, x_n$.

Let $\Delta x = x_i - x_{i-1} = \frac{b-a}{n}$ be the width of each interval.

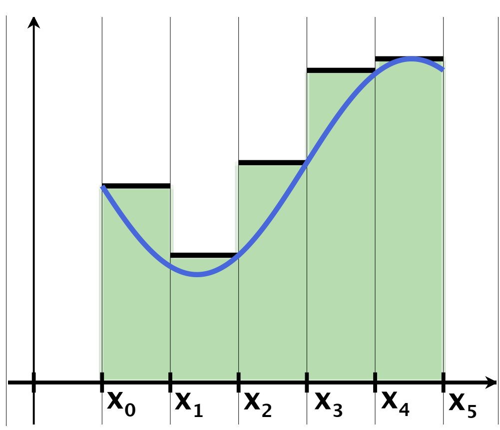
The **Upper Riemann Sum** is: let M_i be the *maximum* value of the function on that i^{th} interval, so

$$U(f, P) = M_1\Delta x + M_2\Delta x + \cdots + M_n\Delta x.$$

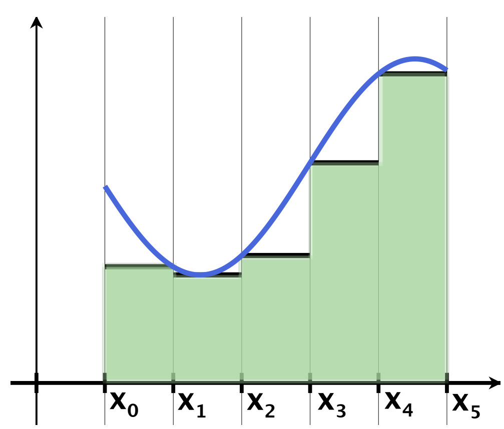
The **Lower Riemann Sum** is: let m_i be the *minimum* value of the function on that i^{th} interval, so

$$L(f, P) = m_1\Delta x + m_2\Delta x + \cdots + m_n\Delta x).$$

Take the limit as $n \to \infty$ or $\Delta x \to 0$.



Upper



Lower

Sigma Notation

If m and n are integers with $m \leq n$, and if f is a function defined on the integers from m to n, then the symbol $\displaystyle\sum_{i=m}^{n} f(i)$, called sigma notation, is means

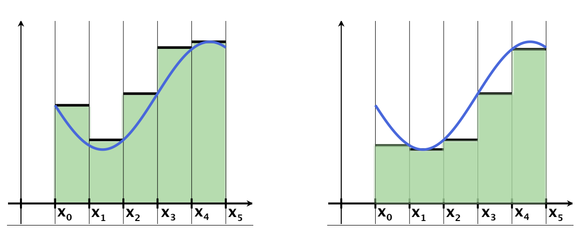
$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n)$$

Examples:
$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$
$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$
$$\sum_{i=1}^{n} \sin(i) = \sin(1) + \sin(2) + \sin(3) + \cdots + \sin(n)$$
$$\sum_{i=0}^{n-1} x^i = 1 + x + x^2 + x^2 + x^3 + x^4 + \cdots + x^{n-1}$$

The Area Problem Revisited

$$\begin{aligned} \text{Upper Riemann Sum} &= \sum_{i=1}^{n} M_i \Delta x \\ \text{Lower Riemann Sum} &= \sum_{i=1}^{n} m_i \Delta x, \end{aligned}$$

where M_i and m_i are, respectively, the maximum and minimum values of f on the ith subinterval $[x_{i-1}, x_i]$, $1 \leq i \leq n$.

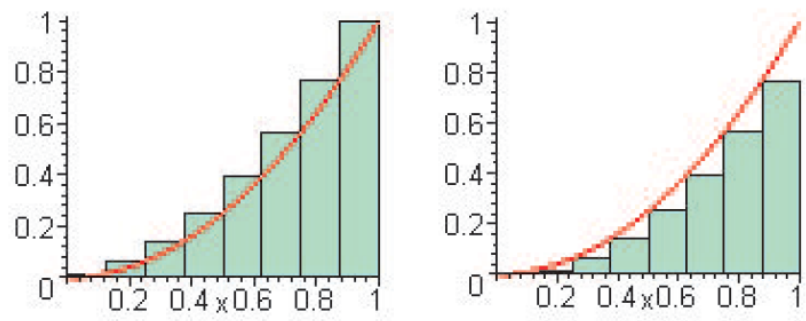


Upper Lower

$\boxed{n = 5}$

Example

Use an Upper Riemann Sum and a Lower Riemann Sum, first with 8, then with 100 subintervals of equal length to approximate the area under the graph of $y = f(x) = x^2$ on the interval $[0, 1]$.



The Definite Integral

We say that f is integrable on $[a, b]$ if there exists a number A such that

$$\text{Lower Riemann Sum} \leq A \leq \text{Upper Riemann Sum}$$

for all n. We write the number as

$$A = \int_a^b f(x)dx$$

and call it the **definite integral** of f over $[a, b]$.

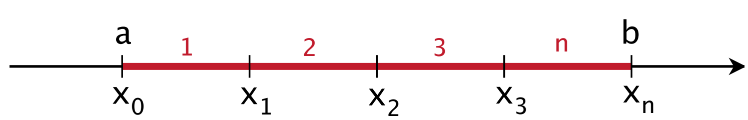
Trickiness: Who wants to find maxima/minima over every interval? Especially as $n \to \infty$? Calculus nightmare!!

More Riemann Sums

Let f be defined on $[a, b]$, and pick a positive integer n. Let

$$\Delta x = \frac{b-a}{n}$$

Notice:



$$x + 0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad x_3 = a + 3\Delta x, \ldots$$

So let

$$\boxed{x_i = a + i * \Delta x.}$$

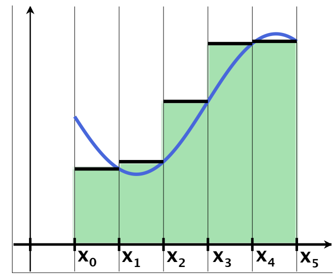
More Riemann Sums

Let f be defined on $[a, b]$, and pick a positive integer n.
Let

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i * \Delta x.$$

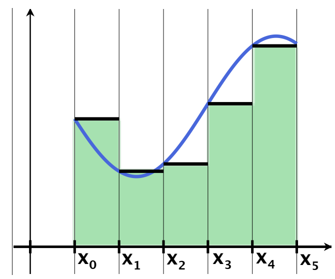
Then the Right Riemann Sum is

$$\sum_{i=1}^{n} f(x_i)\Delta x,$$



and the Left Riemann Sum is

$$\sum_{i=0}^{n-1} f(x_i)\Delta x_i.$$



Integrals made easier

Theorem
If f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Theorem
If f is Riemann integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where c_i is **any** *point in the interval $[x_{i-1}, x_i]$.*

Punchline: (1) Every continuous function has an integral, and (2) we can get there by just using right or left sums! (instead of upper or lower sums)

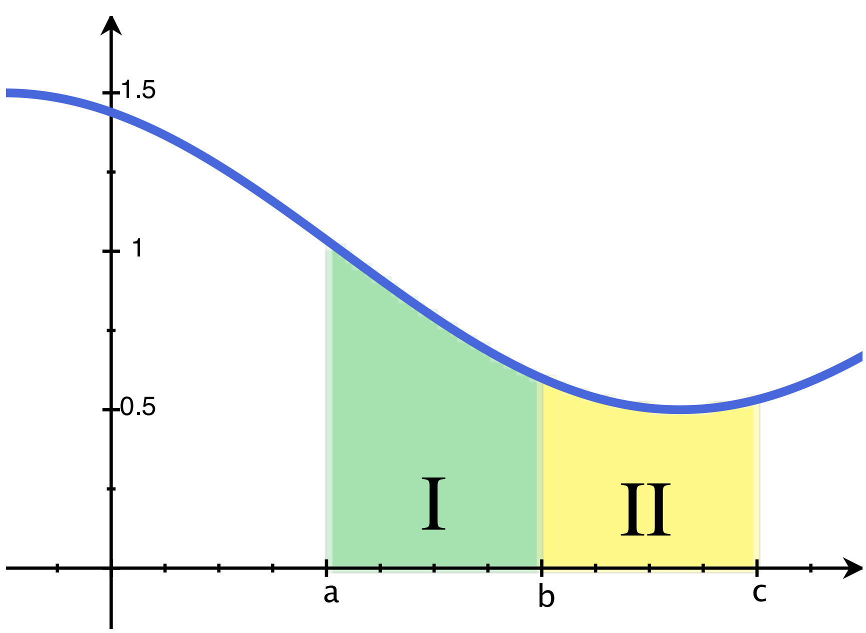
Properties of the Definite Integral

1. $\displaystyle\int_a^a f(x)dx = 0.$

2. If f is integrable and
   (a) $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)dx$ equals the area of the region under the graph of f and above the interval $[a, b]$;
   (b) $f(x) \leq 0$ on $[a, b]$, then $\int_a^b f(x)dx$ equals the **negative** of the area of the region between the interval $[a, b]$ and the graph of f.

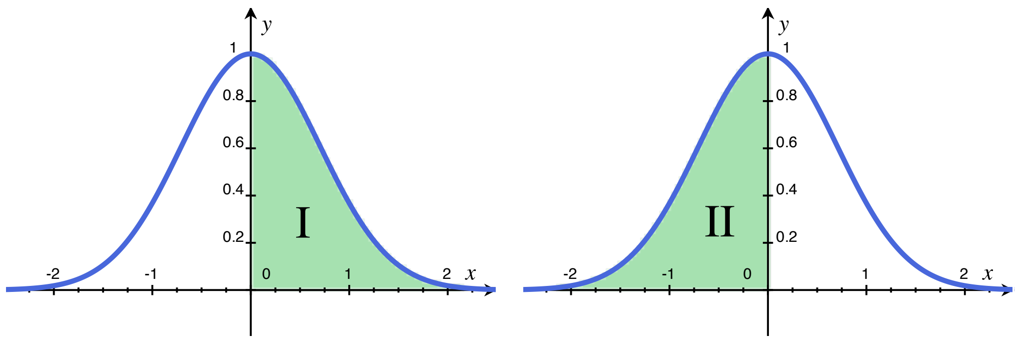
3. $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx.$

4. If $a < b < c$, $\displaystyle\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$



5. If f is an **even** function, then

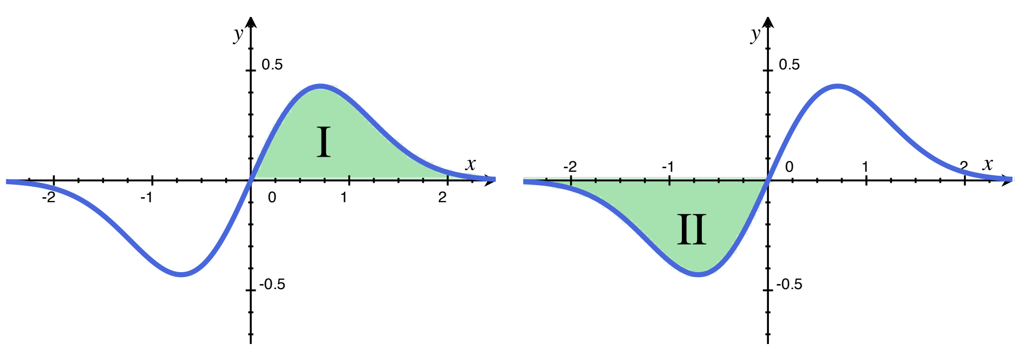
$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$



Area I = Area II

6. If f is an **odd** function, then

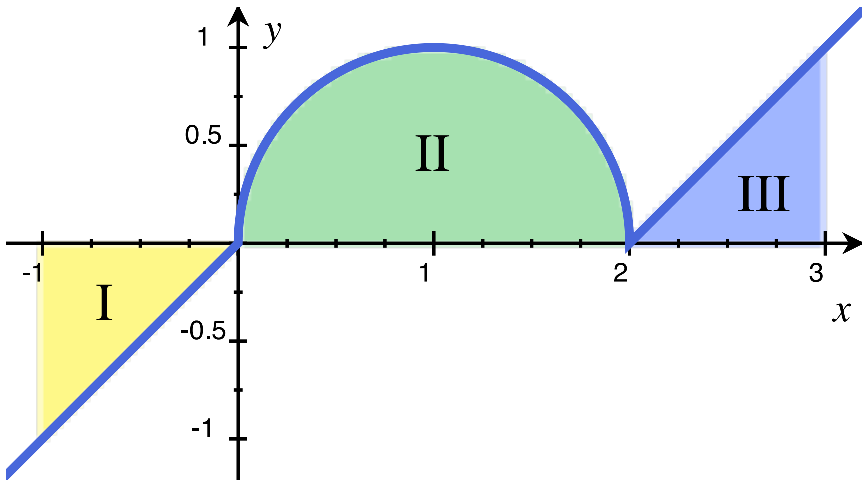
$$\int_{-a}^{a} f(x)dx = 0.$$



Area I = Area II

Example

$$\text{If } f(x) = \begin{cases} x, & x < 0, \\ \sqrt{1-(x-1)^2}, & 0 \geq x \leq 2, \\ x-2, & x \geq 2, \end{cases} \text{ what is } \int_{-1}^{3} f(x)dx?$$



Mean Value Theorem for Definite Integrals

Theorem

Let f be continuous on the interval $[a, b]$. Then there exists c in $[a, b]$ such that

$$\int_a^b f(x)dx = (b-a)f(c).$$

Definition

The *average value* of a continuous function on the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx.$$