Modelling Accumulations

The purpose of calculus is twofold:

1. to find how something is changing, given what it's doing;
2. to find what something is doing, given how it's changing.

We did derivatives

(a) **algebraically** (derivative rules, what is the function?), and

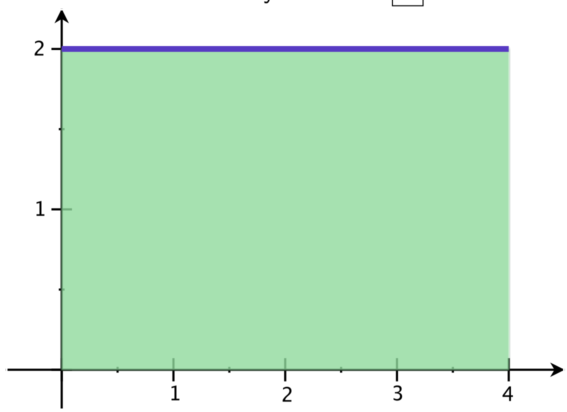
(b) **geometrically** (slopes, increasing/decreasing, what does it look like?)

We did antiderivatives algebraically (what is the function?).
Today: geometric meaning of antiderivatives.

If you travel at 2 mph for 4 hours, how far have you gone?

Answer: 8 miles.

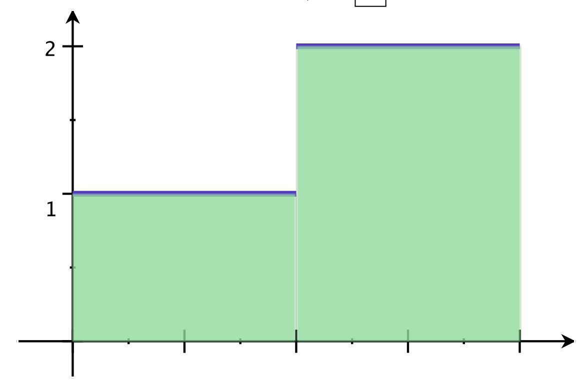
Another way: Area = $\boxed{8}$



(graph of speed, i.e. graph of derivative)

If you travel at 1 mph for 2 hours, and 2 mph for 2 hours, how far have you gone?

Area = 2+4=$\boxed{6}$



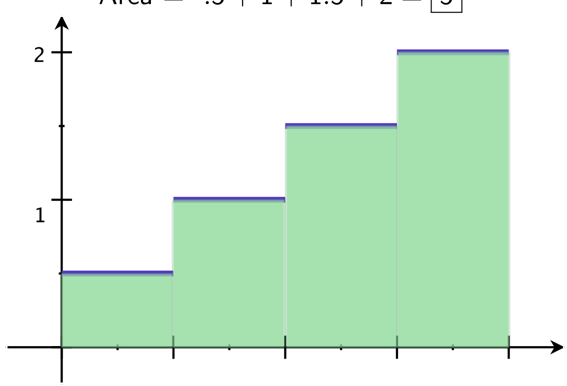
(graph of speed, i.e. graph of derivative)

If you travel at

.5 mph for 1 hour,
1 mph for 1 hour,
1.5 mph for 1 hour,
2 mph for 1 hour,

how far have you gone?

Area = $.5 + 1 + 1.5 + 2 = \boxed{5}$



(graph of speed, i.e. graph of derivative)

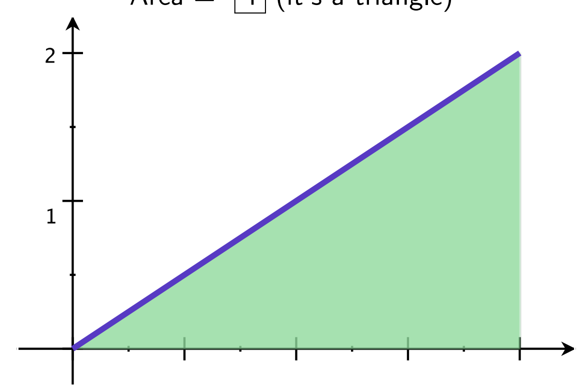
If you travel at $\frac{1}{2}t$ mph for 4 hours, how far have you gone?

Check our answer using antiderivatives from last time:

$$\text{position} = s(t) = \int \frac{1}{2}t\ dt = \frac{1}{4}t^2 + C$$

So distance $= s(4) - s(0) = \frac{1}{4} * 16 + C - (\frac{1}{4} * 0 + C) = 4$ ✓

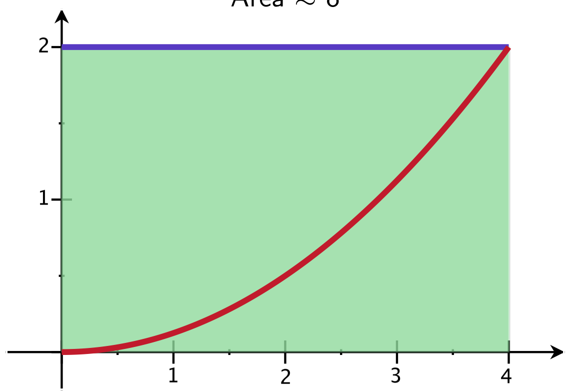
Area = $\boxed{4}$ (it's a triangle)



(graph of speed, i.e. graph of derivative)

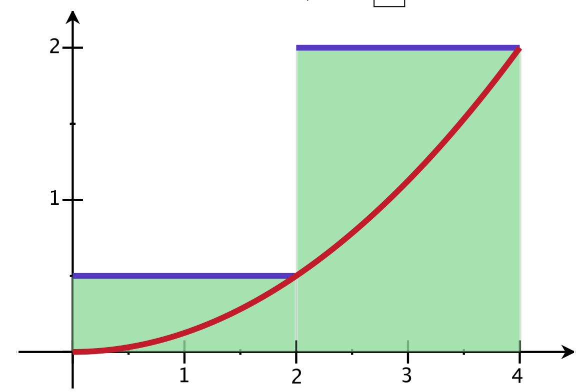
Estimate the area under the curve
$y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

Estimate 1: pick the highest point
Area ≈ 8



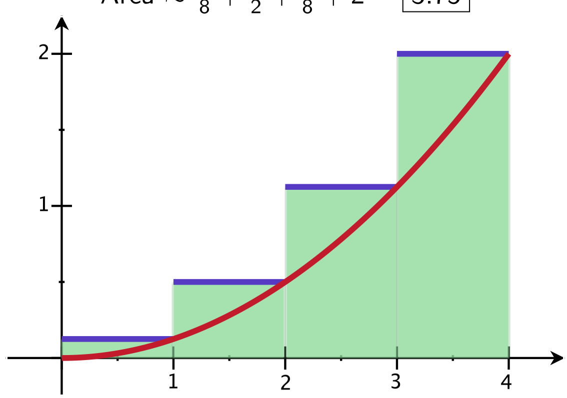
Estimate the area under the curve
$y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

Estimate 2: pick two points
Area $\approx 1{+}4 = \boxed{5}$



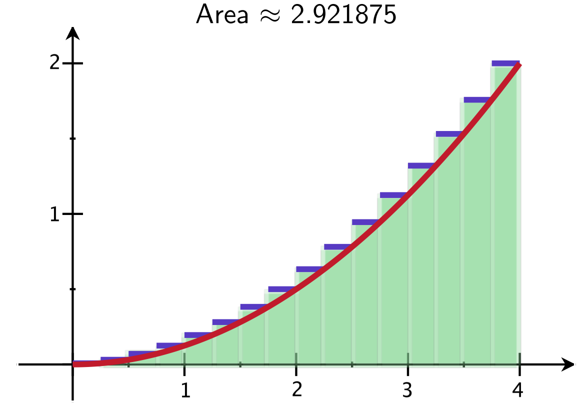
Estimate the area under the curve
$y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

Estimate 3: pick four points
Area $\approx \frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 2 = \boxed{3.75}$



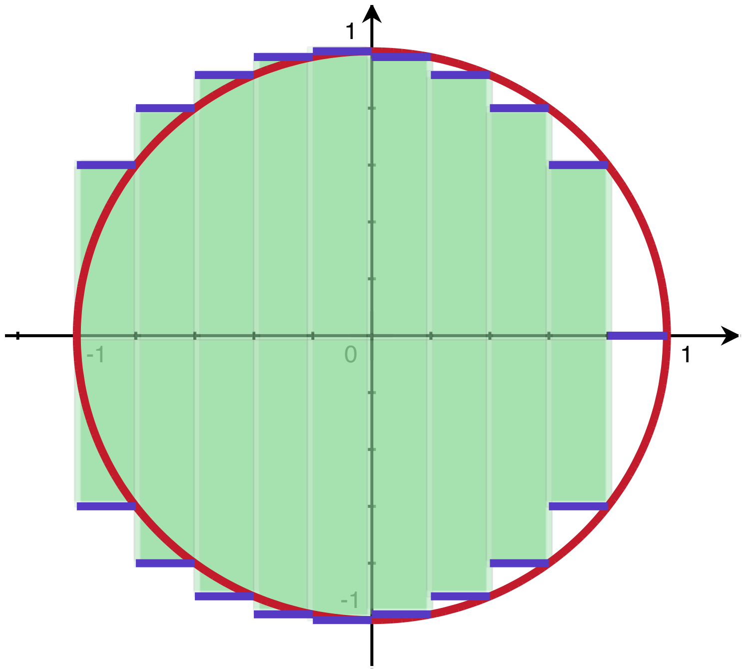
Estimate the area under the curve
$y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

Estimate 5: pick sixteen points
Area ≈ 2.921875



Estimating the Area of a Circle with $r = 1$

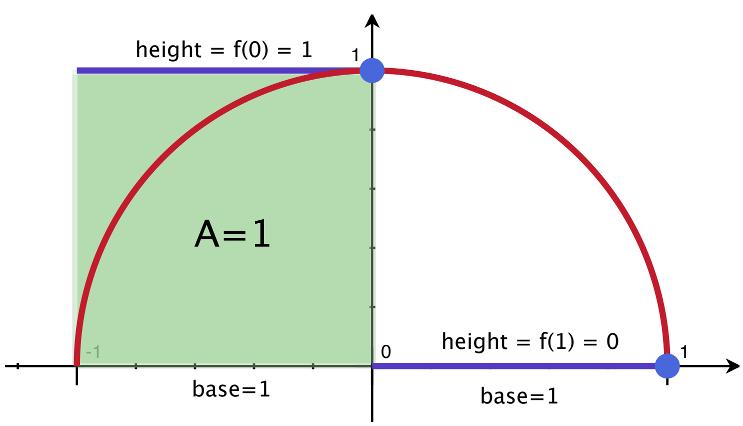
Divide it up into rectangles:



Estimating the Area of a Circle with $r = 1$

Divide it up into rectangles:

Estimate area of the half circle ($f(x) = \sqrt{1 - x^2}$) and mult. by 2.



# rect.	Area
4	2*1 = 2
4*2	
4*3	
4*4	
4*5	

Draw pictures and estimate area for 4*2 and 4*3 rectangles:

Estimating the Area of a Circle with $r = 1$

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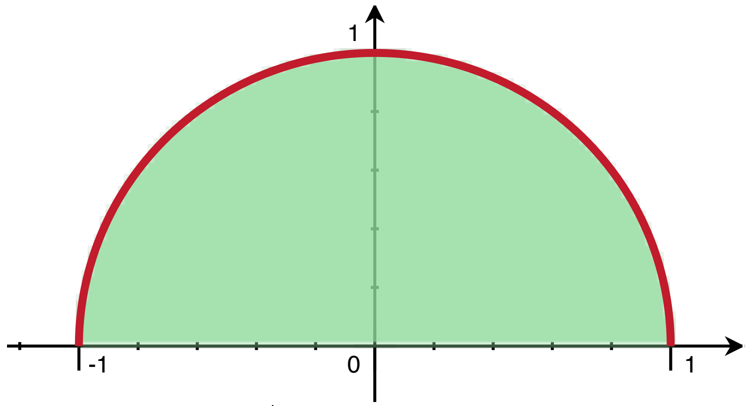


# rect.	Area
4	$2*1 = 2$
4*2	$\sqrt{3} + 1 \approx 2.732$
4*3	$\frac{2}{3} + \frac{8\sqrt{2}+4\sqrt{5}}{9} \approx 2.918$
4*4	2.996
4*5	3.037
4*100	3.140

Numerical Integration

Big idea: Estimating, and then taking a limit.

Let the number of pieces go to ∞
i.e. let the base of the rectangle for to 0.

Good for:

1. Approximating accumulated change when the antiderivative is unavailable.
2. Making precise the notion of 'area' (we'll also to lengths and volumes)

Example: estimating volume using data

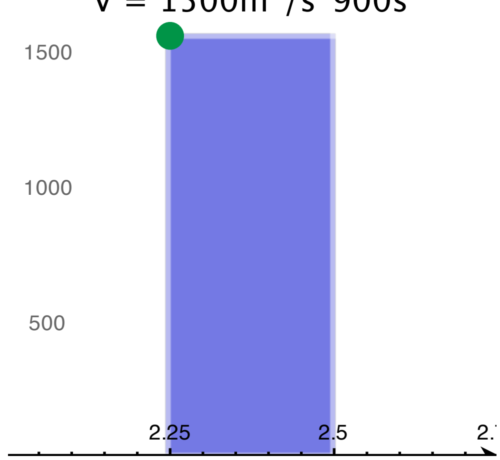
A small dam breaks on a river. The average flow out of the stream is given by the following:

hours	m^3/s	hours	m^3/s	hours	m^3/s
0	150	4.25	1460	8.25	423
0.25	230	4.5	1350	8.5	390
0.5	310	4.75	1270	8.75	365
0.75	430	5	1150	9	325
1	550	5.25	1030	9.25	300
1.25	750	5.5	950	9.5	280
1.5	950	5.75	892	9.75	260
1.75	1150	6	837	10	233
2	1350	6.25	770	10.25	220
2.25	1550	6.5	725	10.5	199
2.5	1700	6.75	658	10.75	188
2.75	1745	7	610	11	180
3	1750	7.25	579	11.25	175
3.25	1740	7.5	535	11.5	168
3.5	1700	7.75	500	11.75	155
3.75	1630	8	460	12	150
4	1550				

Over each time interval, we estimate the volume of water by

$$\text{Average rate} \times 900 \text{ s}$$

V = 1500m^3/s*900s



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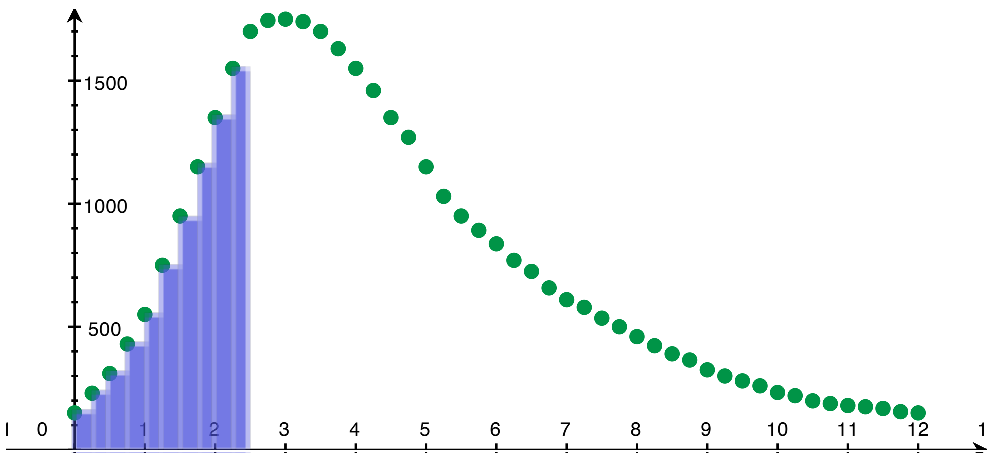


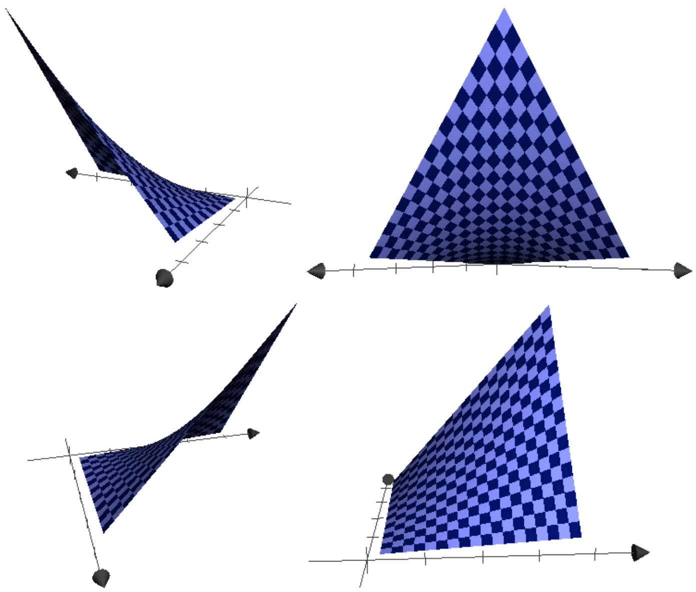
Over each time interval, we estimate the volume of water by

Average rate × 900 s

hours	m^3	hours	m^3	hours	m^3
0	135000	4.25	1314000	8.25	380700
0.25	207000	4.5	1215000	8.5	351000
0.5	279000	4.75	1143000	8.75	328500
0.75	387000	5	1035000	9	292500
1	495000	5.25	927000	9.25	270000
1.25	675000	5.5	855000	9.5	252000
1.5	855000	5.75	802800	9.75	234000
1.75	1035000	6	753300	10	209700
2	1215000	6.25	693000	10.25	198000
2.25	1395000	6.5	652500	10.5	179100
2.5	1530000	6.75	592200	10.75	169200
2.75	1570500	7	549000	11	162000
3	1575000	7.25	521100	11.25	157500
3.25	1566000	7.5	481500	11.5	151200
3.5	1530000	7.75	450000	11.75	139500
3.75	1467000	8	414000	12	135000
4	1395000			total=33,319,800	

Example: estimating volume under a function of 2 variables

A tent is raised and has height given by xy over the 2×2 grid where $0 < x < 2$ and $0 < y < 2$. What is the volume of the tent?



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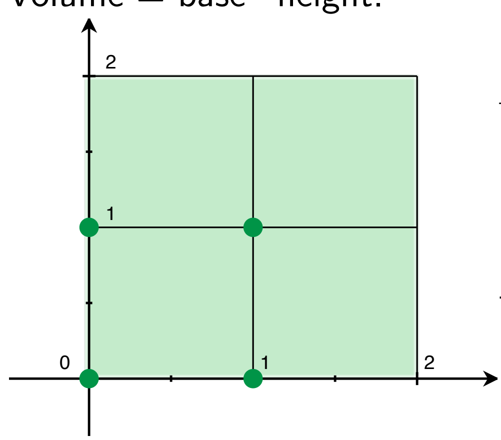
Estimate via boxes!
Volume = base *height.



x	y	height $= xy$	volume
0	0	0	0 * 1
0	1	0	0 * 1
1	0	0	0 * 1
1	1	1	1 * 1

total volume ≈ 1

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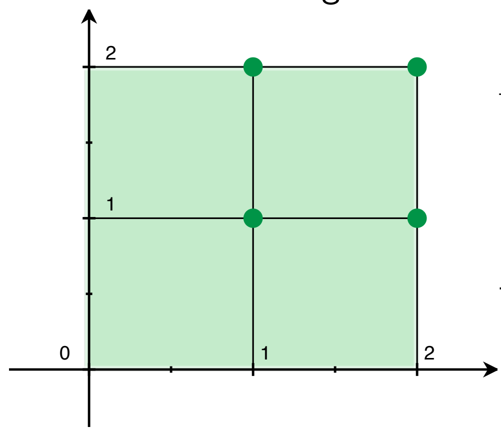

x	y	height $= xy$	volume
1	1	1	1 * 1
1	2	2	2 * 1
2	1	2	2 * 1
2	2	4	4 * 1

total volume ≈ 9

Example: estimating volume under a function of 2 variables

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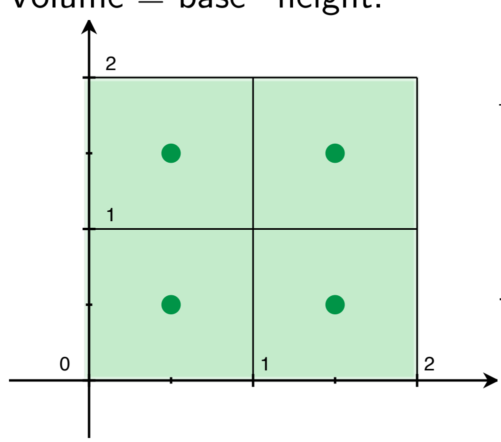


x	y	height $= xy$	volume
.5	.5	.25	.5 * 1
.5	1.5	.75	.75 * 1
1.5	.5	.75	.75 * 1
1.5	1.5	2.25	2.25 * 1

total volume ≈ 4.25

WolframAlpha computational knowledge engine

Enter what you want to calculate or know about:

int e^(-x^2) dx

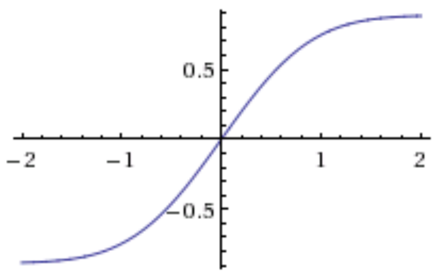
Examples Random

Indefinite integral:

$$\int e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}\,\mathrm{erf}(x) + \text{constant}$$

$\mathrm{erf}(x)$ is the error function »

Plots of the integral:



(x from −2 to 2)

Error function

From Wikipedia, the free encyclopedia

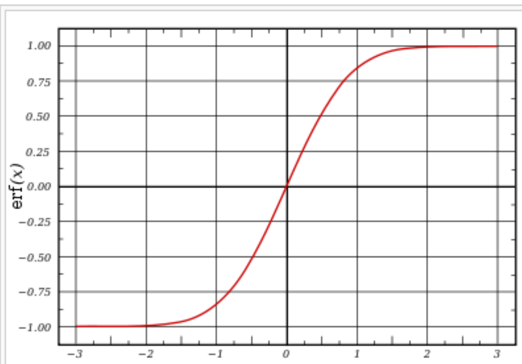
In mathematics, the **error function** (also called the **Gauss error function**) is a special function (non-elementary) of sigmoid shape which occurs in probability, statistics and partial differential equations. It is defined as:[1][2]

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

(When *x* is negative, the integral is interpreted as the negative of the integral from *x* to zero.)

The **complementary error function**, denoted *erfc*, is defined as

$$\begin{aligned}\mathrm{erfc}(x) &= 1 - \mathrm{erf}(x)\\ &= \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt.\end{aligned}$$

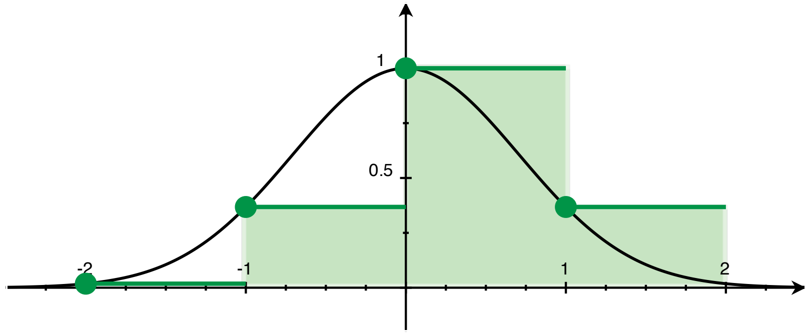


Plot of the error function

Other methods of numerical integration

We did rectangles, but we could use other shapes (that we know how to integrate under) to better represent the shape of the function.

$f(x) = e^{-x^2}$ between $x = -2$ and $x = 2$: $A = 1.76416\ldots$



For n pieces,

for **rectangles** pick n points to draw n rectangles; $A \approx 1.75407\ldots$

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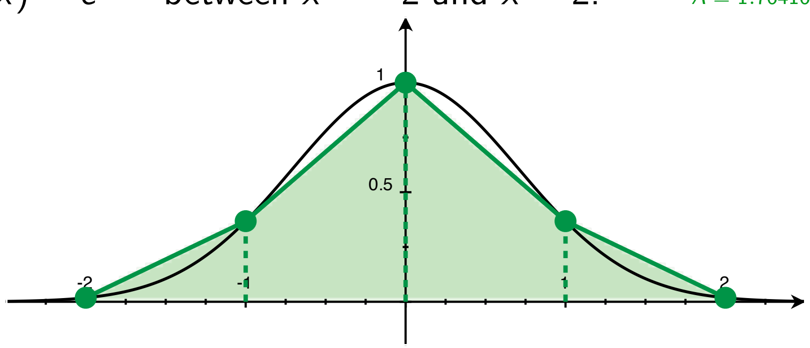

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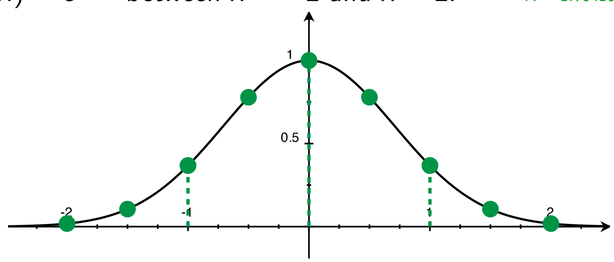
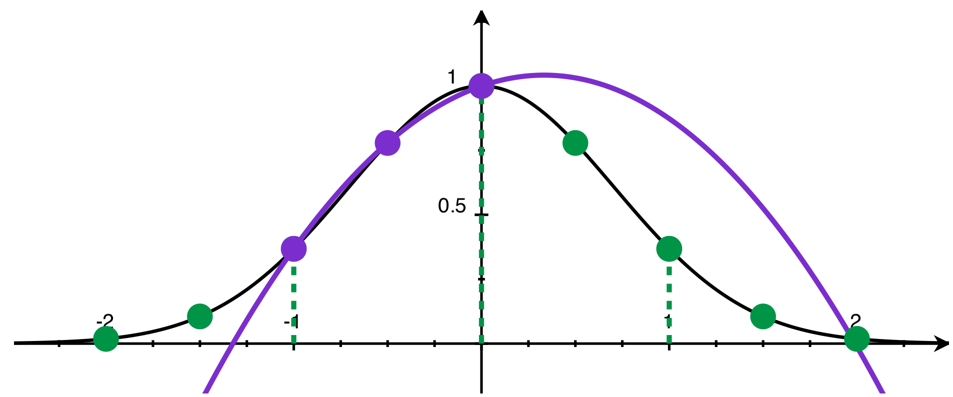




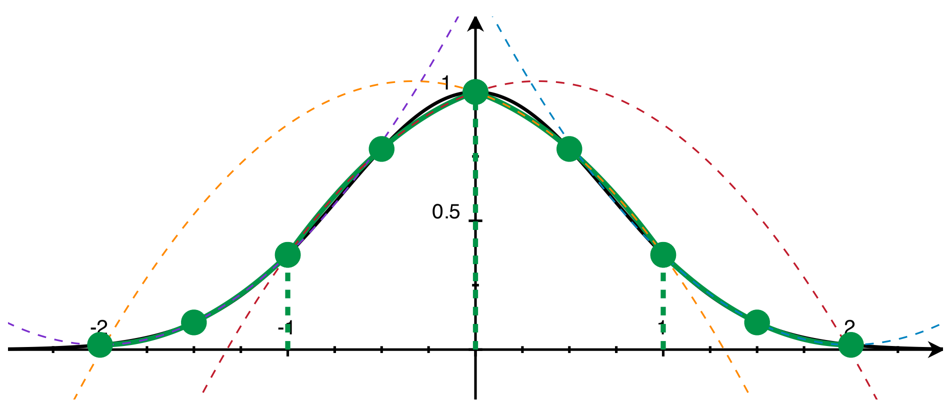
For n pieces,

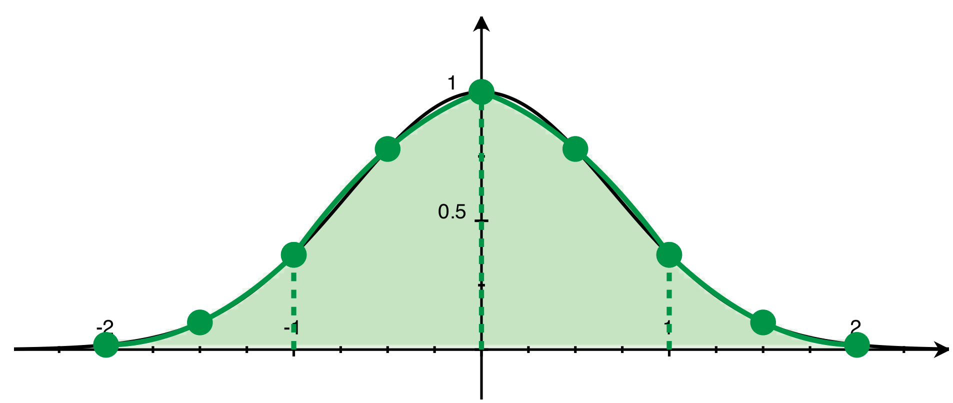
for **rectangles** pick n points to draw n rectangles; $A \approx 1.75407\ldots$

we could also use **trapezoids**, with $n + 1$ points; $A \approx 1.75407\ldots$

we could also use **parabolas**, with $2n + 1$ points;

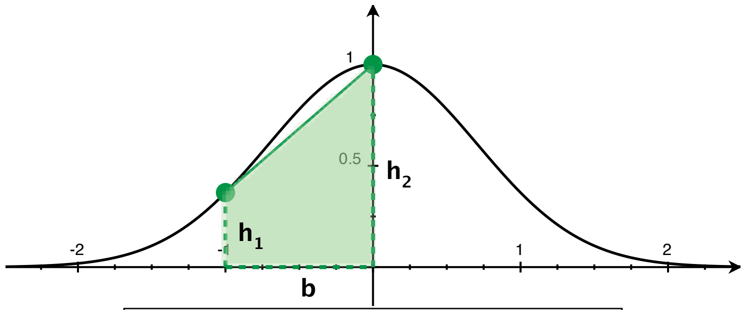
(We call the parabolas **Simpson's rule**)





What you need to know

- For **rectangles**, know how to approximate by hand.
- For **trapezoids**, also know by hand:



$$\boxed{\text{Area (trapeziod)} = b * \frac{h_1+h_2}{2}}$$

For example: $b = 1$, $h_1 = f(-1)$, $h_2 = f(0)$,

so $A_2 = 1 * \frac{f(-1)+f(0)}{2}$

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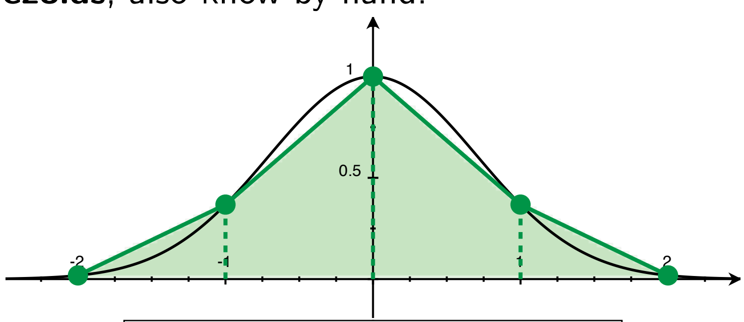

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so $A_2 = 1 * \frac{f(-1)+f(0)}{2}$

$$A = 1 * \frac{f(-2)+f(-1)}{2} + 1 * \frac{f(-1)+f(0)}{2} + 1 * \frac{f(0)+f(1)}{2} + 1 * \frac{f(1)+f(2)}{2}$$

$= \frac{1}{2} * \left[f(-2) + f(2) + 2\big(f(-1) + f(0) + f(1)\big)\right]$ (see book for general form)

- For **Simpson's rule** (parabolas), know how to use applet.

  Warning about conventions: In the book and webwork, n is the number of "subintervals". In class and in the applet, n is the number of parabolas. So if *webwork* says $n = 6$, plug in $n = 3$ to the *applet*.