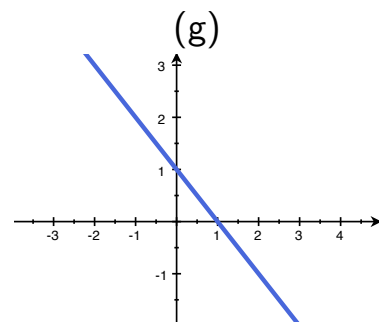
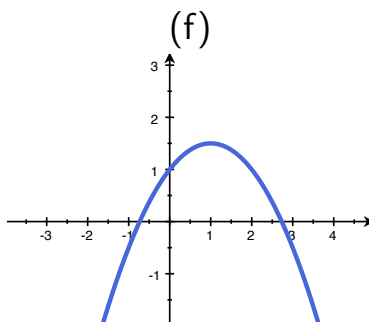
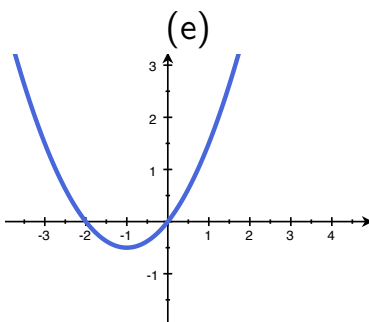
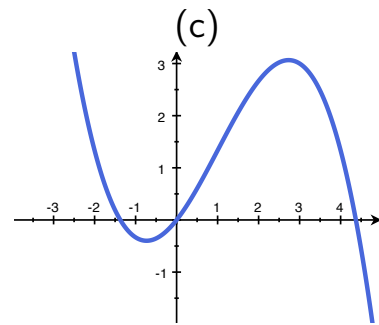
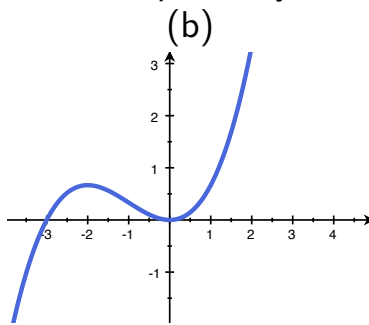
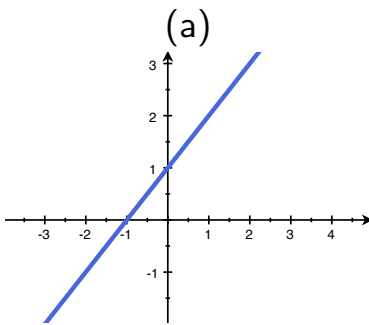


Curve Sketching

Warm up

Below are pictured six functions: $f, f', f'', g, g',$ and g'' . Pick out the two functions that could be f and g , and match them to their first and second derivatives, respectively.

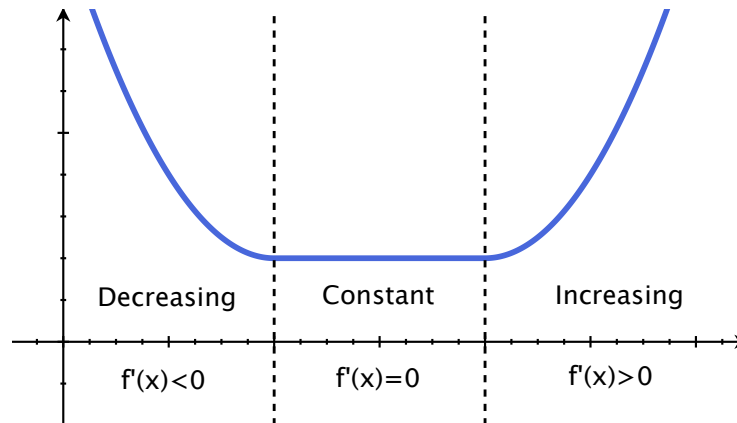


Review: Increasing/Decreasing

Suppose that f is **continuous** on $[a, b]$ and **differentiable** on the open interval (a, b) . Then

If $f'(x)$ is $\begin{cases} \text{positive} \\ \text{negative} \\ \text{zero} \end{cases}$ for every x in (a, b) then f is $\begin{cases} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{cases}$ on $[a, b]$.

What it looks like:



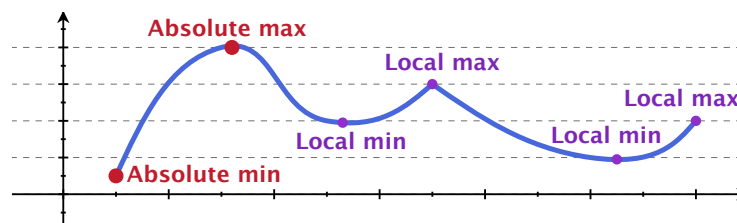
Review: Extreme values

If f is continuous on a closed interval $[a, b]$, then there is a point in the interval where f is largest (**maximized**) and a point where f is smallest (**minimized**).

The maxima or minima will happen either

1. at an endpoint, or
2. at a **critical point**, a point c where $f'(c) = 0$ or $f'(c)$ is undefined.

Vocab: If $f'(c)$ is undefined, c is also called a **singular point**.



Review: finding **absolute** min/max on **closed** intervals

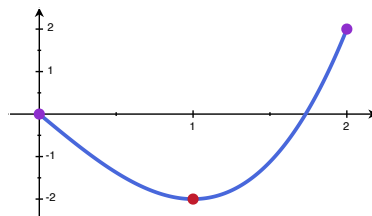
1. Calculate $f'(x)$.
2. Find where $f'(x)$ is 0 or undefined on $[a, b]$ (critical/singular points).
3. Evaluate $f(x)$ at the critical and singular points, and at endpoints. The largest (reps. smallest) value among these is the maximum (reps. minimum).

Example: Let $f(x) = x^3 - 3x$. What are the min/max values on the interval $[0, 2]$.

$$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$$

So $f'(x) = 0$ if $x = -1$ or $\boxed{1}$.

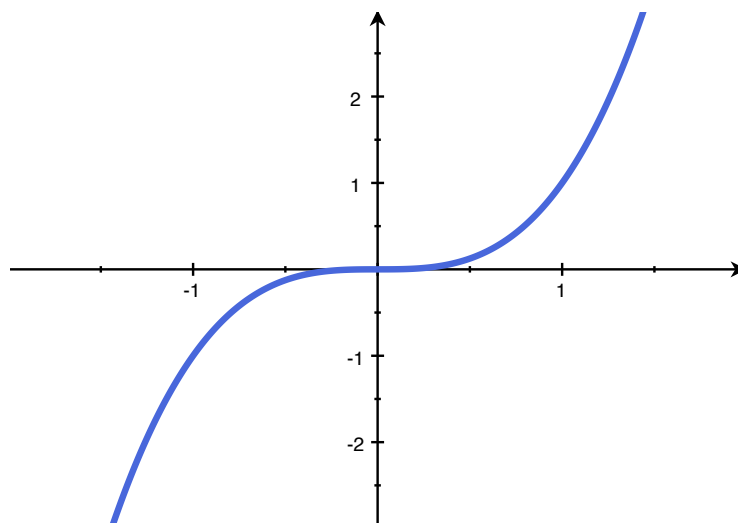
x	$f(x)$	
1	-2	critical points
0	0	end points
2	2	



Finding **local** min/max on **any** intervals

Warning: Not all critical points are local minima or maxima:

Example: If $f(x) = x^3$, then $f'(x) = 3x^2$, and so $f'(0) = 0$:



Finding local extrema: The First Derivative Test

Suppose

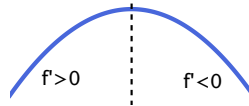
f is **continuous** on (a, b) ,

c is in (a, b) and is a **critical point** of $f(x)$, and

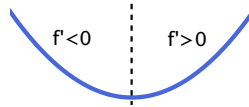
f is **differentiable** on (a, b) (except possibly at $x = c$)

Then the value $f(c)$ can be classified as follows:

1. If $f'(x)$ changes from **positive** \rightarrow **negative** at $x = c$, then $f(c)$ is a **local maximum**.



2. If $f'(x)$ changes from **negative** \rightarrow **positive** at $x = c$, then $f(c)$ is a **local minimum**.



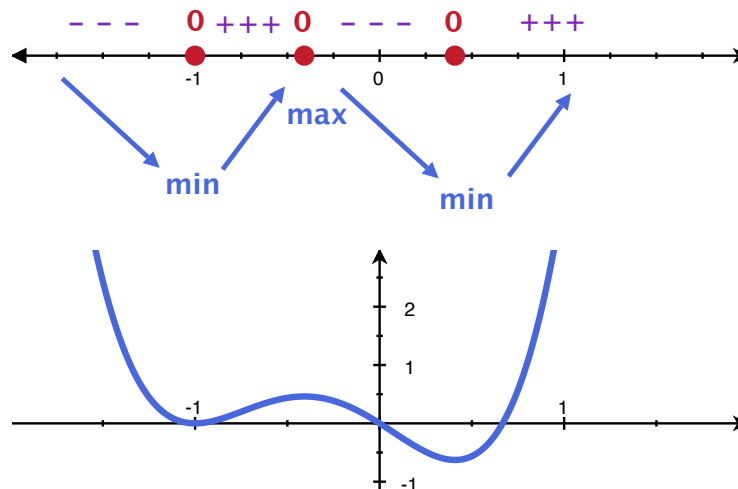
3. If $f'(x)$ doesn't change sign, then it's neither a min or a max.

Example

Find the local extrema of $f(x) = 3x^4 + 4x^3 - x^2 - 2x$ over the whole real line.

Calculate $f'(x)$:

$$f'(x) = 12x^3 + 12x^2 - 2x - 2 = 12(x + 1)(x - 1/\sqrt{6})(x + 1/\sqrt{6})$$



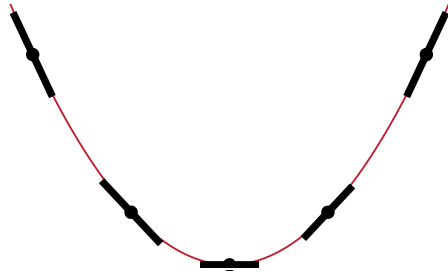
Example

Find the local extrema of $f(x) = \frac{x^4 + 1}{x^2}$ over the whole real line.

[Hint: Make sure to write the derivative like $f'(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.]

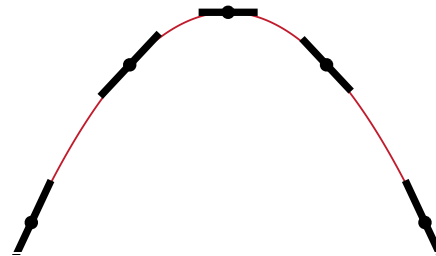
Concavity

Q. How can we measure when a function is concave up or down?



Concave up
 $f'(x)$ is increasing

$$f''(x) > 0$$

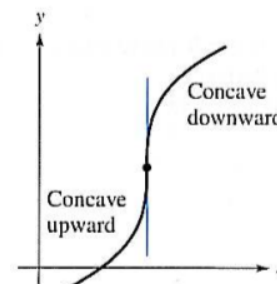
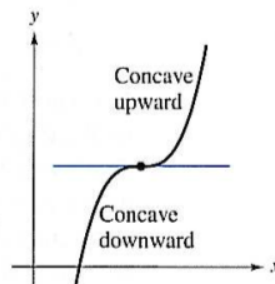
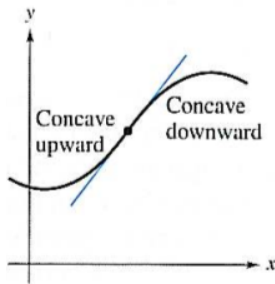


Concave down
 $f'(x)$ is decreasing

$$f''(x) < 0$$

Concavity and Inflection Points

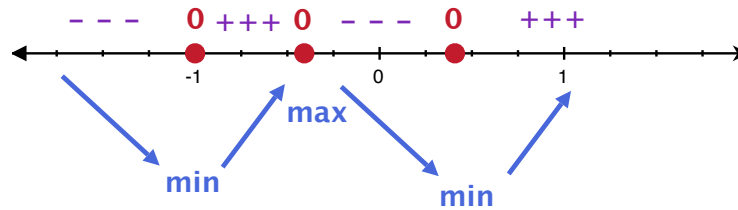
Definition: The function f has an **inflection point** at the point $x = c$ if $f(c)$ exists and the concavity changes at $x = c$ from up to down or vice versa.



Back to the example $f(x) = 3x^4 + 4x^3 - x^2 - 2x$

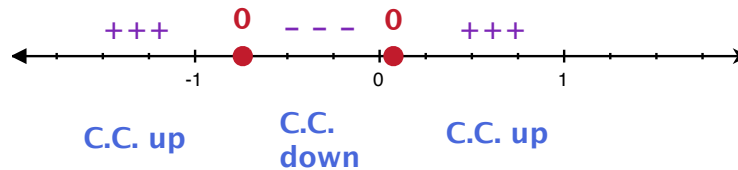
Find the inflection points of $f(x)$, and where $f(x)$ is concave up or down.

We calculated $f'(x) = 12x^3 + 12x^2 - 2x - 2$.

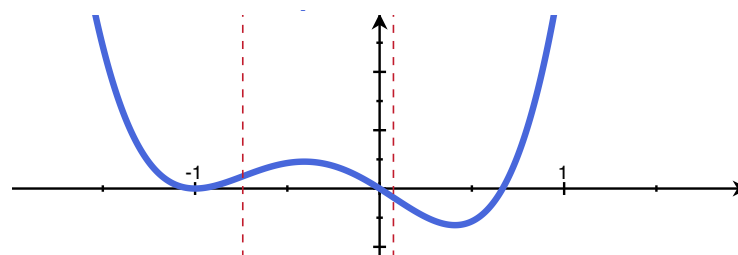
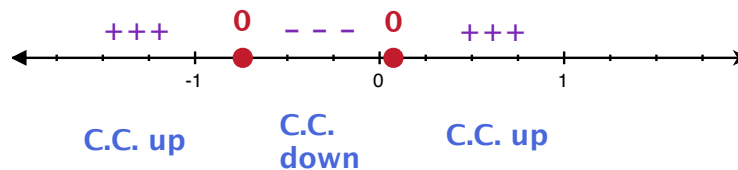
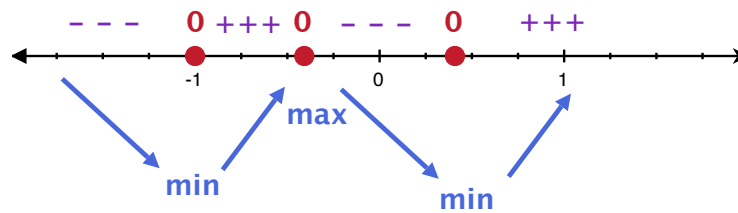


So

$$f''(x) = 36x^2 + 24x - 2 = (6x - (\sqrt{6} + 2))(6x + \sqrt{6} + 2)$$

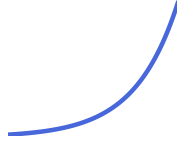


Putting it together

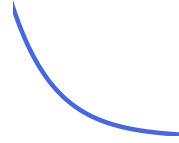


What the pieces look like

concave up
and increasing



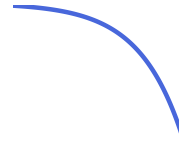
concave up
and decreasing



concave down
and increasing



concave down
and decreasing



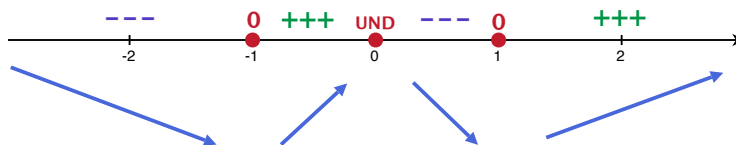
Last elements of graphing

Back to the example where $f(x) = \frac{x^4+1}{x^2}$:

Step 0: Domain. $f(x)$ is defined everywhere except $x = 0$

Step 1: Increasing/decreasing.

We found $f'(x) = \frac{2(x^4-1)}{x^3} = 2 \frac{(x^2+1)(x+1)(x-1)}{x^3}$



Step 2: Concavity.

$$f''(x) = 2 \frac{(4x^3)(x^3) - (x^4 - 1)(3x^2)}{x^6} = 2 \frac{4x^6 - 3x^6 + 3x^2}{x^6} = 2 \frac{x^4 + 3}{x^4}$$

Therefore $f''(x)$ is always positive, but is undef. at 0.

So $f(x)$ is always **concave up**

Last elements of graphing

$f(x) = \frac{x^4+1}{x^2}$ continued...

Step 4: Extreme behavior.

(a) What is $\lim_{x \rightarrow -\infty} f(x)$? What is $\lim_{x \rightarrow \infty} f(x)$?

Are there any *horizontal* asymptotes?

For example, $\lim_{x \rightarrow -\infty} \frac{x^4 + 1}{x^2} = \infty$ $\lim_{x \rightarrow \infty} \frac{x^4 + 1}{x^2} = \infty$,

so there are no horizontal asymptotes.

(b) For any hole in the domain $x = a$, what is $\lim_{x \rightarrow a^-} f(x)$?

$\lim_{x \rightarrow a^+} f(x)$? Are there any *vertical* asymptotes?

For example, $\lim_{x \rightarrow 0^-} \frac{x^4 + 1}{x^2} = \infty$ $\lim_{x \rightarrow 0^+} \frac{x^4 + 1}{x^2} = \infty$,

so there is a two-sided vertical asymptote.

Last elements of graphing

$f(x) = \frac{x^4+1}{x^2}$ continued...

Step 5: Plot salient points.

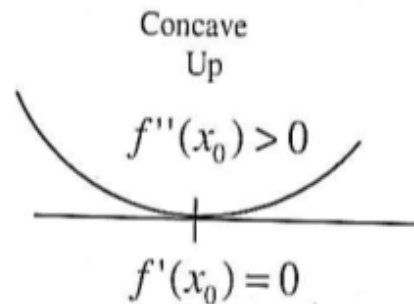
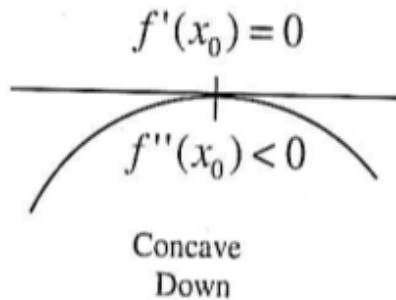
- (a) Find any roots of $f(x)$. (x -intercepts)
- (b) Calculate $f(x)$ at $x = 0$ (y -intercept), critical points, and inflection points.

The second derivative test

Theorem

Let f be a function whose second derivative exists on an interval I containing x_0 .

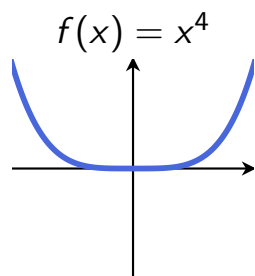
1. If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f(x_0)$ is a local minimum.
2. If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f(x_0)$ is a local maximum.



Warning: If $f'(x_0) = 0$ and $f''(x_0) \leq 0$, then the test **fails**, use the first derivative test to decide.

Why the 2nd derivative test fails when $f''(x_0) = 0$

If $f'(x_0) = 0$ and $f''(x_0) = 0$, anything can happen!

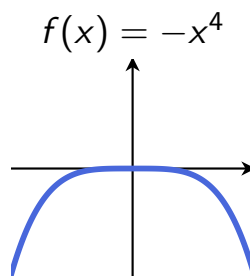


$$f'(x) = 4x^3$$

$$f''(x) = 12x^2$$

$$f'(0) = 0$$

$$f''(0) = 0$$

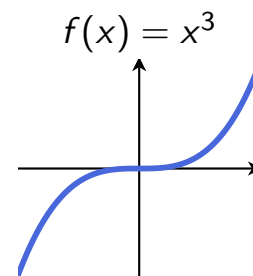


$$f'(x) = -4x^3$$

$$f''(x) = -12x^2$$

$$f'(0) = 0$$

$$f''(0) = 0$$



$$f'(x) = 3x^2$$

$$f''(x) = 6x$$

$$f'(0) = 0$$

$$f''(0) = 0$$

so

(The second derivative being zero just means the function is almost flat.)

Sketch graphs of the following functions:

1. $f(x) = -3x^5 + 5x^3$.

2. $f(x) = \frac{x^2 - 1}{x^2 + 1}$

Instructions:

- * Find any places where $f(x)$ is 0 or undefined.
- * Calculate $f'(x)$ and find critical/singular points.
- * Classify where $f'(x)$ is positive/negative, and therefore where $f(x)$ is increasing/decreasing.
- * Calculate $f''(x)$, and find where it's 0 or undefined.
- * Classify where $f''(x)$ is positive/negative, and therefore where $f(x)$ is concave up/down.
- * Calculate $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ for anything where $f(a)$ is undefined.
- * Calculate $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$ to see what the tails are doing.

Hint for 2: Always simplify as a fraction of polynomials after taking a derivative.

Extra practice: Graphing functions

For each of the following graphing problems also determine (a) where $f(x)$ is defined, (b) where $f(x)$ is continuous, (c) where $f(x)$ is differentiable, (d) where $f(x)$ has vertical asymptotes, (e) if $f(x)$ has horizontal asymptotes, (f) where $f(x)$ is increasing and where it is decreasing, (g) where $f(x)$ is concave up and where it is concave down, (h) what the critical points of $f(x)$ are, and (i) where the points of inflection are. Then sketch a graph.

- Graph $f(x) = a$, where a is a constant.
- Graph $f(x) = ax + b$, where a and b are constants.
- Graph $f(x) = a(x - c) + b$, where a , b and c are constants.
- Graph $f(x) = \begin{cases} 2 - x, & \text{if } x \geq 1, \\ x, & \text{if } 0 \leq x \leq 1. \end{cases}$
- Graph $f(x) = \begin{cases} 2 + x, & \text{if } x \geq 0, \\ 2 - x, & \text{if } x < 0. \end{cases}$
- Graph $f(x) = \begin{cases} 1 - x, & \text{if } x < 1, \\ x^2 - 1, & \text{if } x \geq 1. \end{cases}$
- Graph $f(x) = 2x - x^2$.
- Graph $f(x) = x - x^2 - 27$.
- Graph $f(x) = 3x^2 - 2x - 1$.
- Graph $f(x) = x^3 - x + 1$.
- Graph $f(x) = x^3 - x - 1$.
- Graph $f(x) = (x - 2)^2(x - 1)$.
- Graph $f(x) = 2x^3 - 21x^2 + 36x - 20$.
- Graph $f(x) = 2x^3 + x^2 - 20x$.
- Graph $f(x) = 1 - x^4$.
- Graph $f(x) = x^4 - 3x^2 + x$.
- Graph $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.
- Graph $f(x) = 3x^4 - 16x^3 + 18x^2$.
- Graph $f(x) = 3x^5 - 25x^3 + 60x$.
- Graph $f(x) = x^5 - 4x^4 + 4x^3$.
- Graph $f(x) = x^3(x - 2)^2$.
- Graph $f(x) = (x - 2)^4(x + 1)^3(x - 1)$.
- Graph $f(x) = (x - 3)^5(x + 1)^4$.
- Graph the function $f(x)$ such that $\frac{df}{dx} = 1/x$ and $f(-1) = 2$ and $f(1) = 1$.
- Graph $f(x) = x + 1/x$.
- Graph $f(x) = \frac{x^2 + 2x - 20}{x - 4}$.
- Graph $f(x) = \frac{1}{x^2 + 1}$.
- Graph $f(x) = \frac{x^3}{x^2 + 1}$.
- Graph $f(x) = \frac{x^2 - 1}{x^2 + 1}$.
- Graph $f(x) = \frac{2x^2}{x^2 - 1}$.
- Graph $f(x) = \frac{x^2 + 7x + 3}{x^2}$.
- Graph $f(x) = \frac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4}$.
- Graph $f(x) = \frac{x^2 - 1}{x^3 - 4x}$.