Warm up

Derivative Rules

Use the limit definition of the derivative to calculate the following derivatives.

1. $\frac{d}{dx}(5x+2)$
2. $\frac{d}{dx}(3x-1)$
3. $\frac{d}{dx}\left[(5x+2)+(3x-1)\right]$
4. $\frac{d}{dx}\left[(5x+2)(3x-1)\right]$
5. $\frac{d}{dx}15x^2$
6. $\frac{d}{dx}(15x^2+x-2)$

Remember the power rule says $\frac{d}{dx}x^a = ax^{a-1}$.
Based on your calculations above, which of the following statements seem to be true and which seem to be false?

(a) If you multiply a function $f(x)$ by a number c and then take a derivative, you get the same thing as taking the derivative $'f(x)$ and then multiplying by c. (try comparing 5 to the power rule)

(b) If you add two functions $f(x)$ and $g(x)$ and take a derivative, you get the same answer as taking the derivatives $f'(x)$ and $g'(x)$ and then adding those together. (try comparing 1-3, and then 6 to the power rule)

(c) If you multiply two functions $f(x)$ and $g(x)$ and take a derivative, you get the same answer as taking the derivatives $f'(x)$ and $g'(x)$ and then multiplying those together. (try comparing 1, 2, and 4)

Multiplying by constants: what's going on?

Take another look at $f(x) = 15x^2$. Before, we just expanded and canceled, and were surprised to find something nice happened:

$$\frac{d}{dx}\left[15x^2\right] = \lim_{h\to 0} \frac{15(x+h)^2 - 15x^2}{h}$$

$$= \lim_{h\to 0} \frac{15x^2 + 30xh + 15h^2 - 15x^2}{h}$$

$$= \lim_{h\to 0} 30x + \underbrace{15h}_{\to 0} = \boxed{30x}$$

Let's try again, only pay closer attention to that 15:

$$\frac{d}{dx} x^2 = \lim_{h\to 0} \frac{15(x+h)^2 - 15x^2}{h} = \lim_{h\to 0} 15 * \frac{((x+h)^2 - x^2)}{h}$$

one of our limit rules!

$$= 15 * \underbrace{\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}}_{= \frac{d}{dx} x^2}$$

But now suppose you have any differentiable function $f(x)$ and a number c. [Think: $f(x) = x^2$ and $c = 15$]. Then *in general*

$$\frac{d}{dx}\left(c * f(x)\right) = \lim_{h\to 0} \frac{c * f(x+h) - c * f(x)}{h}$$

$$= \lim_{h\to 0} c * \frac{(f(x+h) - f(x))}{h}$$

limit rule

$$= c * \lim_{h\to 0} \frac{(f(x+h) - f(x))}{h} = c * \frac{d}{dx} f(x)$$

Theorem (Scalars)

If $y = f(x)$ is a differentiable function and c is a constant, then

$$\frac{d}{dx}(c * f(x)) = c * \frac{d}{dx} f(x).$$

Example

Since $\frac{d}{dx}x^2 = 2x$, we have $\frac{d}{dx}15x^2 = 15 * 2x = 30x$.

Taking sums: what's going on?

Take another look at $f(x) = (5x+2) + (3x-1)$. Before, we just simplified first, and were surprised:

$$\frac{d}{dx}\left[(5x+2)+(3x-1)\right] \underset{\text{simplify first}}{=} \frac{d}{dx}\left[8x+1\right] = \lim_{h\to 0} \frac{8(x+h)+1-(8x+1)}{h}$$

$$= \lim_{h\to 0} \frac{8\cancel{h}}{\cancel{h}} = \boxed{8}$$

Let's try again, only pay closer attention to either part of the sum:

go together! go together!

$$\frac{d}{dx}\left((5x+2)+(3x-1)\right) = \lim_{h\to 0} \frac{(5(x+h)+2)+(3(x+h)-1)-[5x+2+3x-1]}{h}$$

$$= \lim_{h\to 0} \frac{[(5(x+h)+2)-(5x+2)] + [(3(x+h)-1)-(3x-1)]}{h}$$

because $\frac{a+b}{c} = \frac{a}{c}+\frac{b}{c}$

$$= \lim_{h\to 0} \frac{[5(x+h)+2]-(5x+2)}{h} + \frac{[3(x+h)-1]-(3x-1)}{h}$$

limit rule!

$$\overset{*}{=} \lim_{h\to 0} \frac{5(x+h)+2-(5x+2)}{h} + \lim_{h\to 0} \frac{3(x+h)-1-(3x-1)}{h}$$

$$= \frac{d}{dx}(5x+2) + \frac{d}{dx}(3x-1)$$

Now, suppose you have any differentiable functions $f(x)$ and $g(x)$ [Think: $f(x) = x^2$ and $c = 15$]. Then *in general*

go together! go together!

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \lim_{h\to 0} \frac{(f(x+h)+g(x+h)) - (f(x)+g(x))}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right]$$

(because $\frac{a+b}{c} = \frac{a}{c}+\frac{b}{c}$)

limit rule!

$$\overset{*}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Theorem (Sums)

If f and g are differentiable functions, then

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$$

Example

Use the three rules we have so far

$$\frac{d}{dx}x^a = ax^{a-1}, \quad \frac{d}{dx}c * f(x) = c * \left(\frac{d}{dx}f(x)\right),$$

$$\text{and} \quad \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

to calculate the derivatives:

1. $\frac{d}{dx}(x^3 - 7x^2 + 6x^{-15})$

2. $\frac{d}{dx}\left(\sqrt{x} + 100\sqrt[17]{x^3} - \frac{3}{x^{19}}\right)$

[hint: rewrite everything from 2 as powers before you do anything]

Products: What's going on?

Take another look at $f(x) = (5x + 2) * (3x - 1)$. Before, we just simplified first, and were... not surprised:

$$\frac{d}{dx}\left[(5x+2)(3x-1)\right] = \frac{d}{dx}\left(15x^2 + x - 2\right)$$

$$= \lim_{h\to 0} \frac{15(x+h)^2 + (x+h) - 2 - (15x^2 + x - 2)}{h}$$

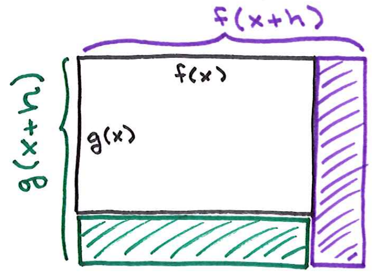
$$= \lim_{h\to 0} \frac{1}{h}\left(15x^2 + \underline{\underline{30xh}} + \underline{\underline{15h^2}} + x + \underline{\underline{h}} - 2 - 15x^2 - x + 2\right)$$

$$= \lim_{h\to 0} 30x + \underline{15h} + 1 = \boxed{30x + 1}$$

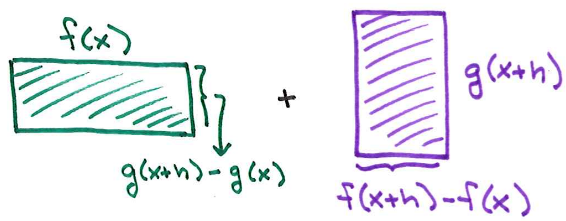
We *didn't* get that the derivative of the products is the product of the derivatives! So what *is* going on here?

To understand how to deal with products, we're going to have to unpack the formula

$$\frac{d}{dx} f(x) * g(x) = \lim_{h \to 0} \frac{f(x+h) * g(x+h) - f(x) * g(x)}{h}$$



$f(x+h) * g(x+h) - f(x) * g(x) =$



$$= f(x) * \left(g(x+h) - g(x)\right) + g(x+h) * \left(f(x+h) - f(x)\right)$$

So

$$\frac{d}{dx} f(x) * g(x) = \lim_{h \to 0} \frac{f(x+h) * g(x+h) - f(x) g(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(f(x) * [g(x+h) - g(x)] + g(x+h)[f(x+h) - f(x)] \right)$$

$$= \lim_{h \to 0} \left[f(x) * \left(\frac{g(x+h) - g(x)}{h} \right) + g(x+h) * \left(\frac{f(x+h) - f(x)}{h} \right) \right]$$

$$= f(x) * \boxed{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}} \to g'(x)$$

$$+ \underbrace{\left(\lim_{h \to 0} g(x+h) \right)}_{g(x)} * \underbrace{\left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right)}_{f'(x)}$$

$$= f(x) * g'(x) + g(x) * f'(x)$$

Theorem (Products)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

Example: Calculate $\frac{d}{dx}\big((5x+2)(3x-1)\big)$:

$$\frac{d}{dx}\big(\underset{f\,\uparrow}{(5x+2)}\underset{g\,\uparrow}{(3x-1)}\big) = \underset{f}{(5x+2)} \cdot \underset{g'}{3} + \underset{g}{(3x-1)} \cdot \underset{f'}{5} = \boxed{\text{30x+1}}$$ ☺

Last rule: Compositions.

Example: Calculate $\frac{d}{dx}(5x+2)^{100}$.
If $f(x) = x^{100}$ and $g(x) = 5x+2$, then $f(g(x)) = (5x+2)^{100}$.
So since $f'(x) = 100x^{99}$ and $g'(x) = 5$, if everything were right and just in the world, we would hope that

$$\frac{d}{dx}(5x+2)^{100} = 100(5)^{99}$$

But it's not!! $\boxed{\frac{d}{dx}(5x+2)^{100} \neq 100(5)^{99}}$

Theorem (Chain rule)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}((f \circ g)(x)) = \frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

Last rule: Compositions.

We won't prove this identity, but we can kind of see where it's coming from:

$$\frac{d}{dx} f(g(x)) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} * \frac{g(x+h) - g(x)}{h}$$

how f changes versus $g(x)$ (instead of versus x) $= f'(g(x))$

$g'(x)$

In Leibniz notation:

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg} * \frac{dg}{dx}$$

$$\text{Chain rule:} \quad \frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Example

Calculate $\frac{d}{dx}(5x+2)^{100}$.

Here,

$$f(x) = x^{100} \quad \text{and} \quad g(x) = 5x + 2.$$

So

$$f'(x) = 100x^{99} \quad \text{and} \quad g'(x) = 5$$

$$\text{Chain rule:} \quad \frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Example

Calculate $\frac{d}{dx}\left(\sqrt{x^7+5}\right)$.

Here,

$$f(x) = \sqrt{x} = x^{1/2} \quad \text{and} \quad g(x) = x^7 + 5.$$

So

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad g'(x) = 7x^6$$

and so

$$\frac{d}{dx}\left(\sqrt{x^7+5}\right) = \frac{1}{2\sqrt{x^7+5}} \cdot 7x^6.$$

Derivative rules

In summary, the derivative rules we have now are

1. **Power rule:** $\frac{d}{dx}x^a = ax^{a-1}$
2. **Scalar rule:** $\frac{d}{dx}c * f(x) = c * \frac{d}{dx}f(x)$
3. **Sum rule:** $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
4. **Product rule:** $\frac{d}{dx}(f(x) * g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$
5. **Chain rule:** $\frac{d}{dx}f(g(x)) = f'(g(x)) * g'(x)$

Examples

Use everything you know to calculate the derivatives of

1. $(3x^2 + x + 1)(5x + 1)$
2. $(3x^2 + x + 1)(5x + 1)^2$
3. $(5x + 1)^{10}$
4. $(3x^2 + x + 1)(5x + 1)^{10}$
5. $\dfrac{\sqrt{x^2 - x}}{x + x^{-1}}$
6. $\dfrac{1}{\sqrt[3]{x^2 + 7x^{1/2}}}$

Use the derivative rules (not limits) to prove the identities

a. **Reciprocal identity**: $\dfrac{d}{dx}\dfrac{1}{f(x)} = -\dfrac{f'(x)}{f^2(x)}$

b. **Quotient identity**: $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

c. **Many products identity**:

$$\begin{aligned}\frac{d}{dx}&(f(x) * g(x) * h(x) * k(x)) \\ &= \Big(f(x)g(x)h(x)\Big) * k'(x) + \Big(f(x)g(x)k(x)\Big) * h'(x) \\ &\quad + \Big(f(x)h(x)k(x)\Big) * g'(x) + \Big(g(x)h(x)k(x)\Big) * f'(x)\end{aligned}$$