Exponential and Logarithmic Functions

The Basics

If n and m are positive integers...

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n} \qquad \text{(WeBWoRK: } a\wedge n \text{ or } a**n\text{)}$$

Some identities:

$$a^n * a^m = a^{n+m} \qquad (a^n)^m = a^{n*m}$$

(Notice: a^{m^n} means $a^{(m^n)}$, since $(a^m)^n$ can be written another way)

$$a^n * b^n = (a*b)^n$$

Examples:

$$2^5 = 2*2*2*2*2$$

$$2^5 * 2^3 = (2*2*2*2*2)*(2*2*2) = 2^8$$

$$(2^3)^5 = (2*2*2)*(2*2*2)*(2*2*2)*(2*2*2)*(2*2*2) = 2^{15}$$

$$2^{3^5} = 2^{243} >> (2^3)^5 = 2^{15}$$

$$2^3 * 5^3 = (2*2*2)*(5*5*5) = (2*5)*(2*5)*(2*5) = (2*5)^3$$

Pushing it further...

Take for granted: If n and m are positive integers,

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n}, \qquad a^n * a^m = a^{n+m}, \qquad (a^n)^m = a^{n*m}.$$

Notice:

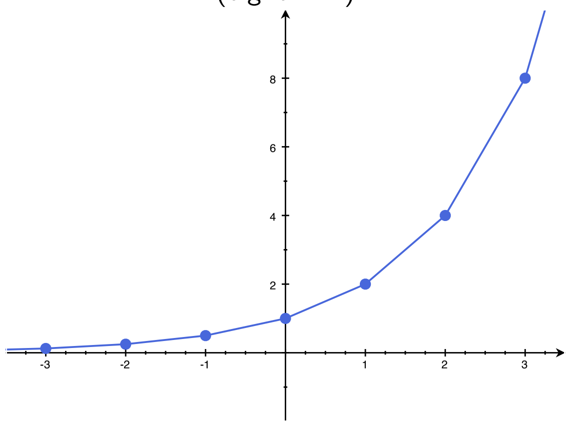
1. What is a^0?
$$a^n = a^{n+0} = a^n * a^0, \qquad \text{so } \boxed{a^0 = 1}.$$
2. What is a^x if x is negative?
$$a^n * a^{-n} = a^{n-n} = a^0 = 1, \qquad \text{so } \boxed{a^{-n} = 1/(a^n)}.$$
3. What is a^x if x is a fraction?
$$(a^n)^{1/n} = a^{n*\frac{1}{n}} = a^1 = a, \qquad \text{so } \boxed{a^{1/n} = \sqrt[n]{a}}$$
$$\text{and } \boxed{a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m}.$$

Example: $8^{5/3} = \left(\sqrt[3]{8}\right)^5 = 2^5 = 32$ or $8^{5/3} = \sqrt[3]{8^5} = \sqrt[3]{32{,}768} = 32$

What is a^x for all x?

If $a > 1$:

(e.g. $a = 2$)



$x = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

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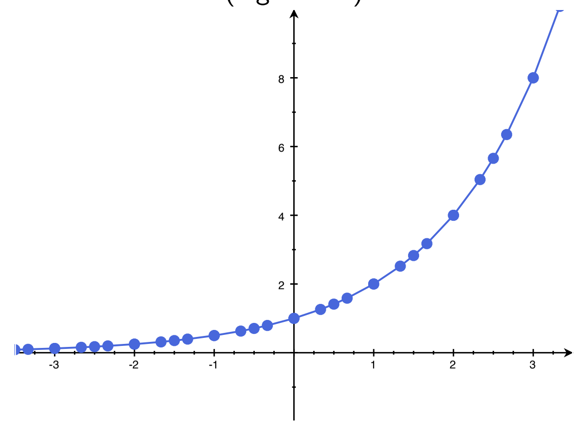

$x = n/2$ and $n/3$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

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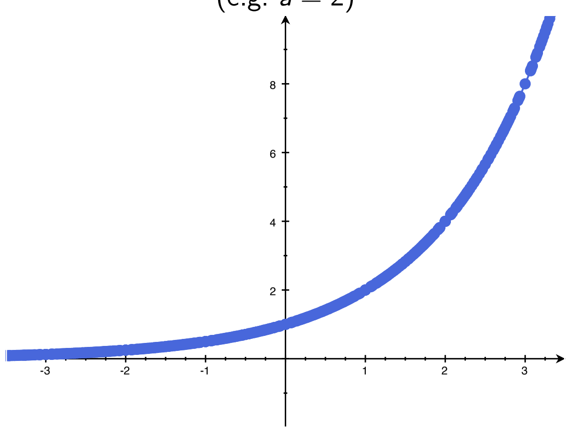

$x = n/2,\ n/3,\ \ldots,\ n/15$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

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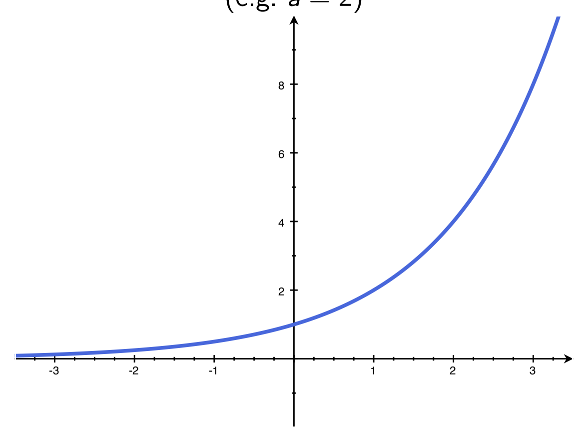
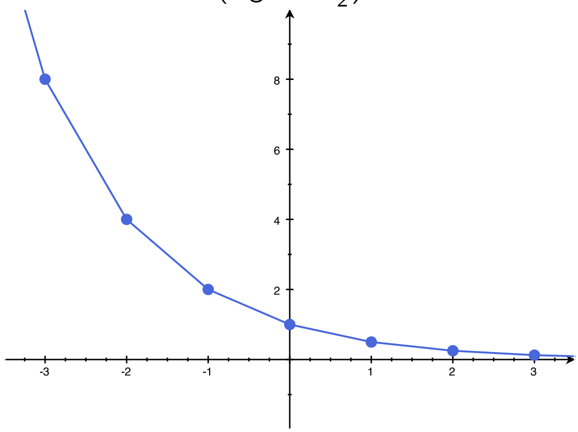


$y = a^x$

What is a^x for all x?

If $0 < a < 1$:

(e.g. $a = \frac{1}{2}$)



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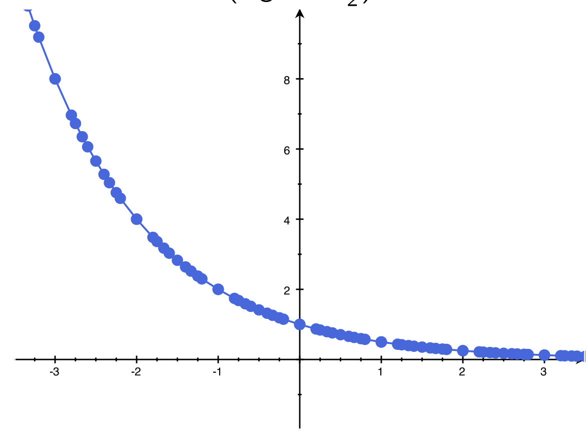

$x = n/2, n/3, n/4, n/5$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

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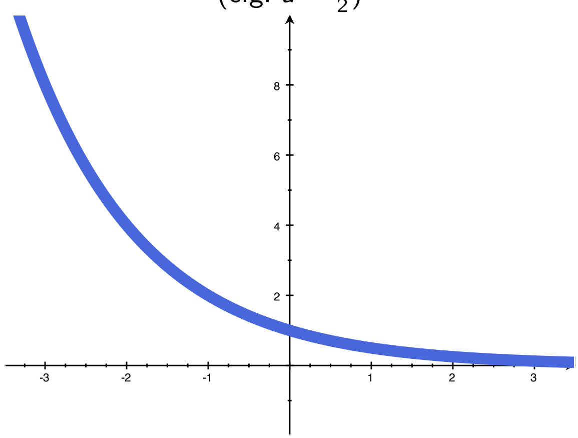
(e.g. $a = \frac{1}{2}$)



$x = n/2,\ n/3,\ \ldots,\ n/100$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

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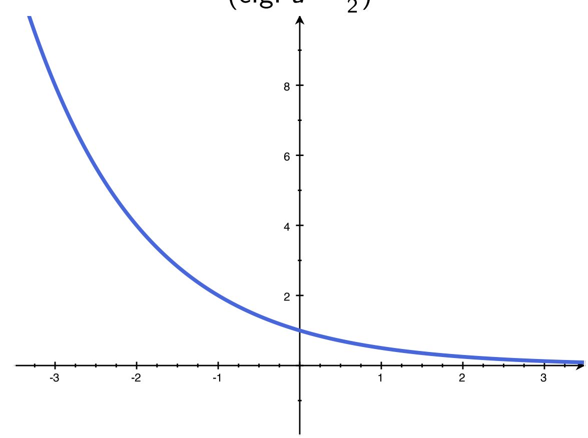
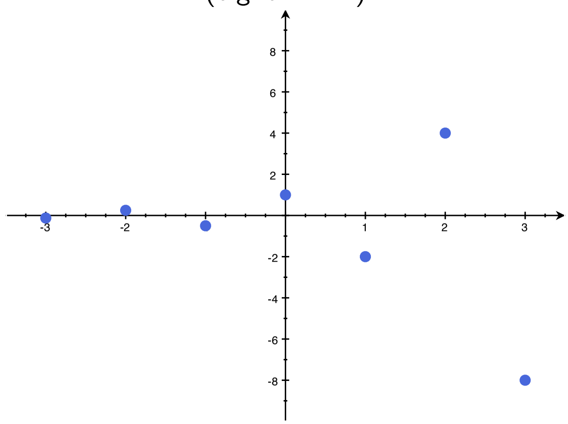


$y = a^x$

What is a^x for all x?

If $0 > a$:

(e.g. $a = -2$)



$x = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

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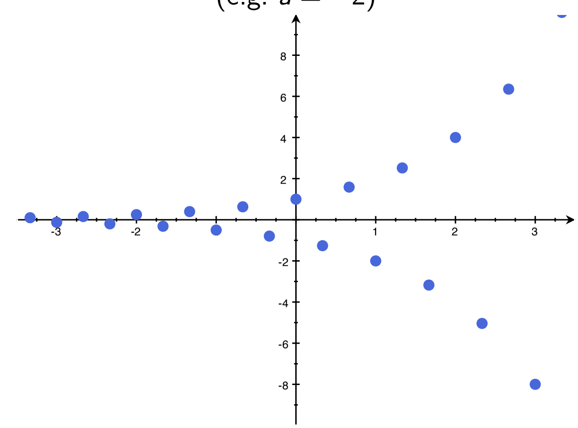

$x = n/3$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

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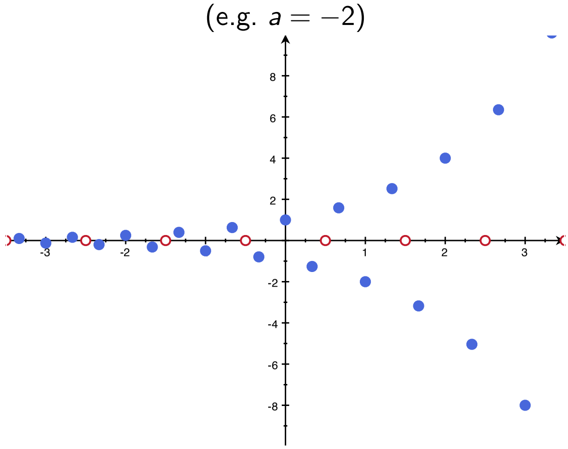

$x = n/3$ and $n/2$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

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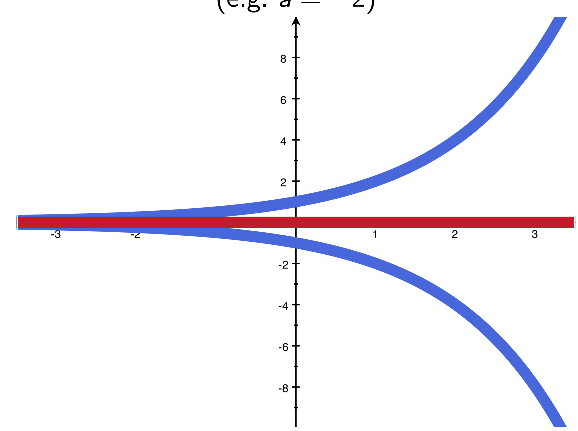
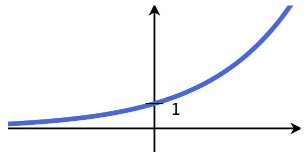


$x = n/2,\ n/3,\ \ldots,\ n/100$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

OH NO!

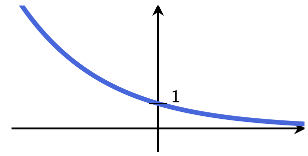
The function a^x:

$1 < a$:



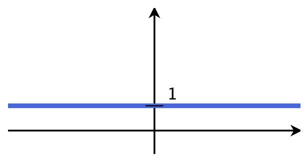
D: $(-\infty, \infty)$, R: $(0, \infty)$

$0 < a < 1$:



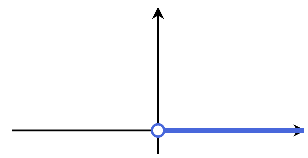
D: $(-\infty, \infty)$, R: $(0, \infty)$

$a = 1$:



D: $(-\infty, \infty)$, R: $\{1\}$

$a = 0$:



D: $(0, \infty)$, R: $\{0\}$

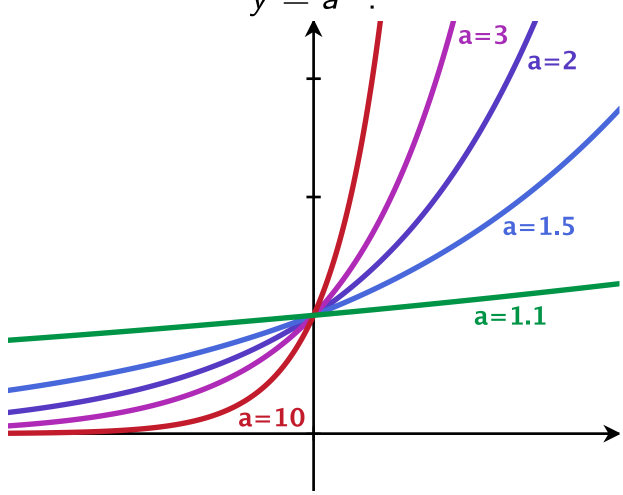
Properties:

$a^b * a^c = a^{b+c}$ $\qquad (a^b)^c = a^{b*c}$ $\qquad a^{-x} = 1/a^x$ $\qquad a^c * b^c = (ab)^c$

Our favorite exponential function:

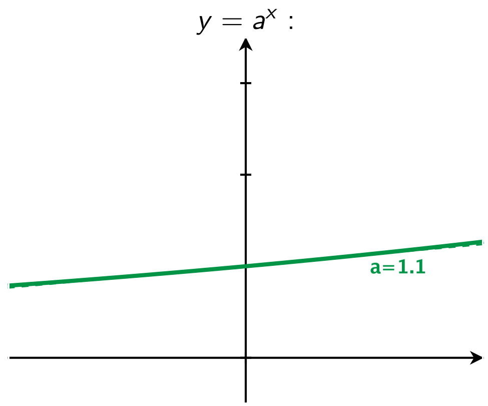
Look at how the function is increasing through the point $(0, 1)$:

$y = a^x$:



Our favorite exponential function:

Look at how the function is increasing through the point $(0, 1)$:



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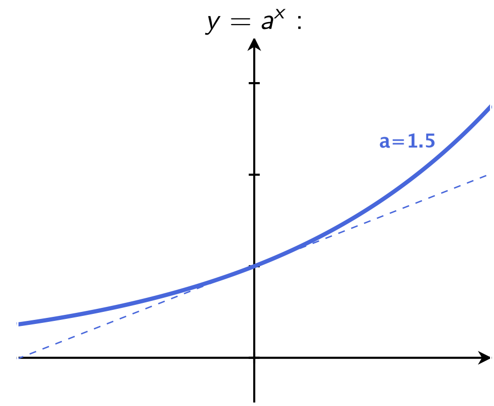

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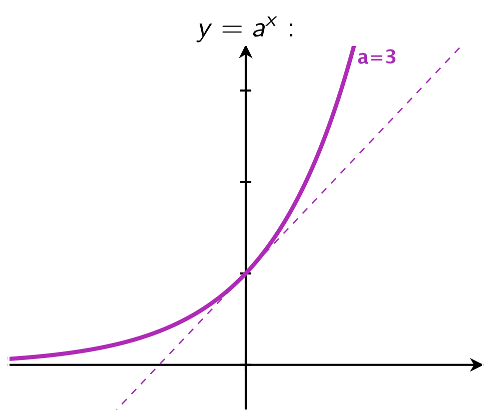

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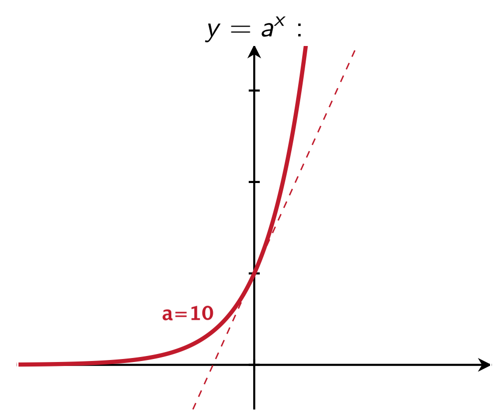

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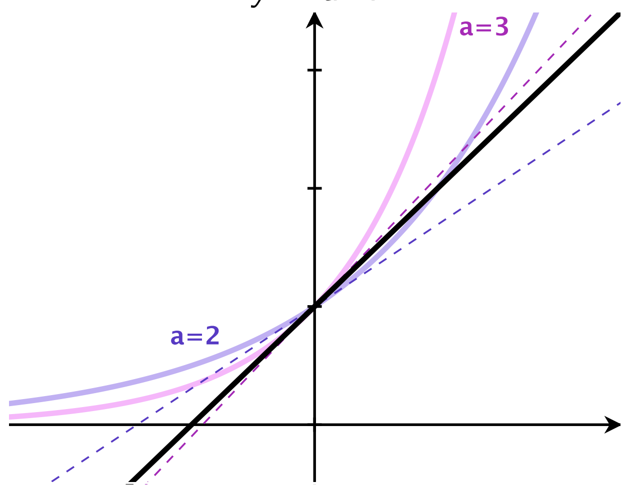

Q: Is there an exponential function whose slope at (0,1) is 1?

Our favorite exponential function:

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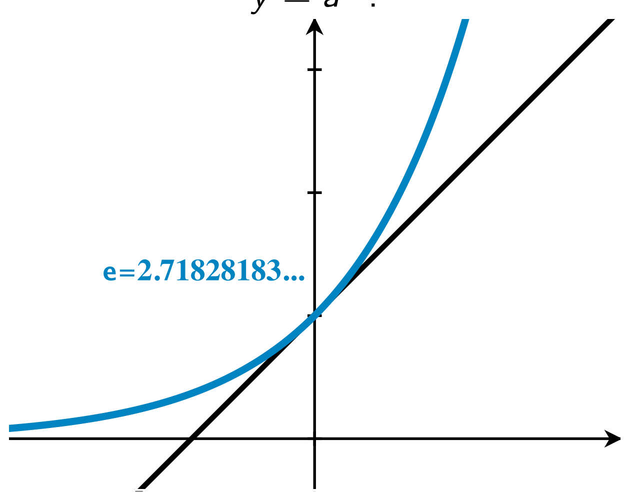
$y = a^x$:



Q: Is there an exponential function whose slope at (0,1) is 1?

A: e^x is the exponential function whose slope at (0,1) is 1.

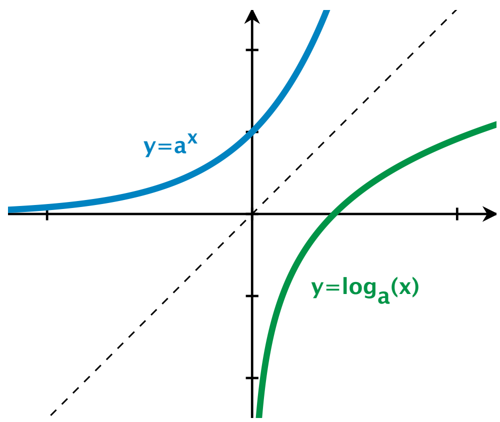
($e = 2.71828183\ldots$ is to calculus as $\pi = 3.14159265\ldots$ is to geometry)

Logarithms

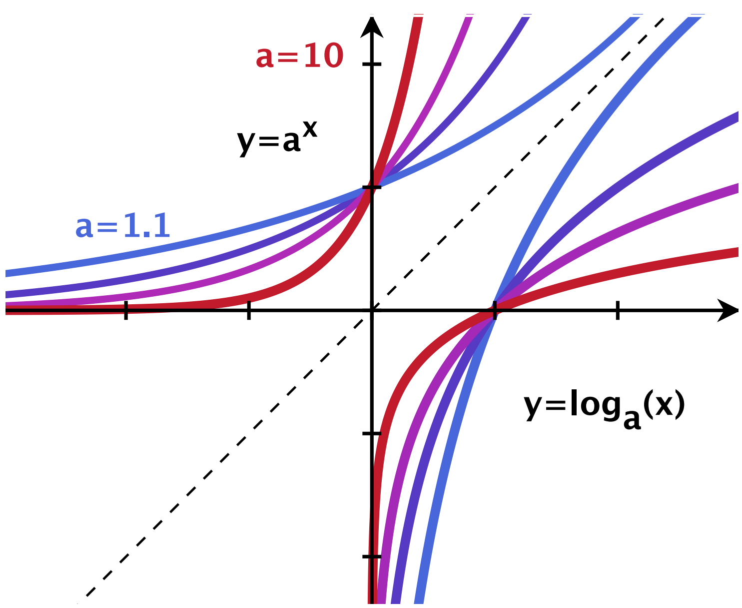
The exponential function a^x has inverse $\log_a(x)$, i.e.

$$\log_a(a^x) = x = a^{\log_a(x)}, \quad \text{i.e.}$$

$$y = a^x \quad \text{if and only if} \quad \log_a(y) = x.$$



Properties of Logarithms



Domain: $(0, \infty)$ i.e. all $x > 0$ Range: $(-\infty, \infty)$ i.e. all x

Properties of Logarithms

Since...

1. $a^0 = 1$
2. $a^1 = a$
3. $a^b * a^c = a^{b+c}$
4. $(a^b)^c = a^{b*c}$

we know...

1. $\log_a(1) = 0$
2. $\log_a(a) = 1$
3. $\log_a(b*c) = \log_a(b) + \log_a(c)$
4. $\log_a(b^c) = c\log_a(b)$

Example: why $\log_a(b*c) = \log_a(b) + \log_a(c)$:

Suppose $y = \log_a(b) + \log_a(c)$.

Then $a^y = a^{\log_a(b)+\log_a(c)} = a^{\log_a(b)}a^{\log_a(c)} = b*c$.

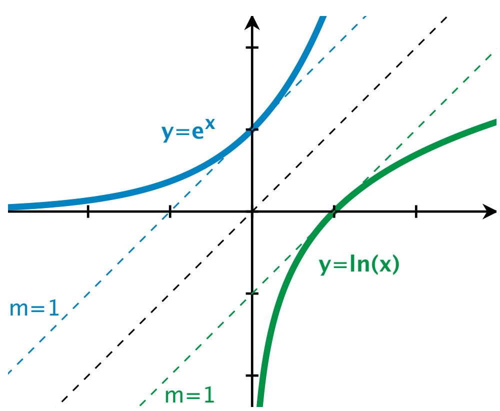
So $y = \log_a(b*c)$ as well!

$$\text{Lastly: } \frac{\log_a(b)}{\log_a(c)} = \log_c(b)$$

Favorite logarithmic function

Remember: $y = e^x$ is the function whose slope through the point (0,1) is 1.
The *inverse* to $y = e^x$ is the *natural log*:

$$\ln(x) = \log_e(x)$$



We will often use the facts that $e^{\ln(x)} = x$ (for $x > 0$) and $\ln(e^x) = x$ (for all x)

Two super useful facts:

Explain why:

(1) $\log_a(b) = \ln(b)/\ln(a)$

(2) $a^b = e^{b\ln(a)}$ [hint: start by rewriting $b\ln(a)$, and use the fact that $e^{\ln(x)} = x$]

Examples:

(1) Condense the logarithmic expressions

$\frac{1}{2}\ln(x)+3\ln(x+1)$ $\quad$ $2\ln(x+5)-\ln(x)$ $\quad$ $\frac{1}{3}(\log_3(x)-\log_3(x+1))$

(2) Solve the following expressions for x:

$e^{-x^2} = e^{-3x-4}$ $\quad$ $3(2^x) = 24$

$2(e^{3x-5}) - 5 = 11$ $\quad$ $\ln(3x+1) - \ln(5) = \ln(2x)$