Warm-up

Suppose $f(x)$ is a differentiable function.
Then what are (in terms of x and $f(x)$ and $f'(x)$)

1. $\frac{d}{dx}\left(f(x) * x\right)$

2. $\frac{d}{dx}\left(f(x) + \sin(x)\right)$

3. $\frac{d}{dx}\left(f(x)/x\right)$

4. $\frac{d}{dx}\left(\left(f(x)\right)^2\right)$

5. $\frac{d}{dx}\left(1+\left[x * \left(f(x)\right)^4\right]\right)$

6. $\frac{d}{dx}\left(\cos\left(x * e^{f(x)}\right)\right)$

1. $\frac{d}{dx}\left(f(x) * x \right) = f'(x) * x \quad + f(x) * 1$

$= \boxed{x f'(x) + f(x)}$

2. $\frac{d}{dx}\left(f(x) + \sin(x)\right) = f'(x) + \cos(x)$

3. $\frac{d}{dx}\left(\frac{f(x)}{x}\right) = \frac{x f'(x) - f(x)}{x^2}$

4. $\frac{d}{dx}\left(\left(f(x)\right)^2\right) = 2 f(x) f'(x)$

5. $\frac{d}{dx}\left[1 + \left(x\left(f(x)\right)^4\right)\right] = 0 + f^4(x) + 4x f^3(x) f'(x)$

6. $\frac{d}{dx}\left(\cos\left(x e^{f(x)}\right)\right) = -\sin\left(x e^{f(x)}\right) * \left[e^{f(x)} + x f'(x) e^{f(x)}\right]$

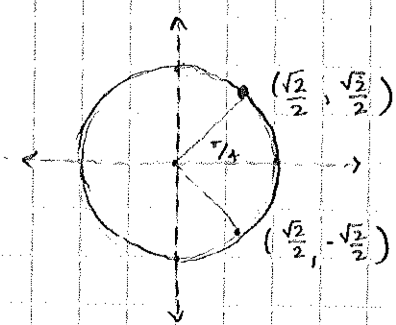
Implicit Differentiation 01/27/2012

Class Notes

Implicit Functions: may not be a function but is a curve that can be expressed as $F(x,y) = c$.

ex. unit circle $x^2 + y^2 = 1$

Calculus so far: solve for y $y = \pm\sqrt{1 - x^2}$



→ What is $\frac{dy}{dx}$ @ $x = \sqrt{2}/2$
- need $y = \sqrt{2}/2$
- take $y = \sqrt{1 - x^2}$ and diffentiate

→ Or $\frac{dy}{dx}$ @ $x = \sqrt{2}/2$, $y = -\sqrt{2}/2$
- look at a different function

Instead use "implicit differentiation":

→ Goal: given $F(x,y) = c$
calculate $\frac{dy}{dx}$ in terms of x and y

→ Strategy: think of y as a function of x
$y = f(x)$

What is $\frac{d}{dx}\left(x^2 + f^2(x)\right)$?
$= 2x + 2f(x) \cdot f'(x)$

What is $\frac{d}{dx} 1$?
$= 0$

If I want to find $\frac{dy}{dx}$ where y and x are related by $x^2 + y^2 = 1$. Start by taking $\frac{d}{dx}$ of both sides.

$$\frac{d}{dx} 1 = 0 = \frac{d}{dx}\left(x^2 + y^2\right)$$
$$= 2x \cdot 2 \cdot y \cdot \boxed{\frac{dy}{dx}}$$

Chain rule in terms of y, b/c don't know what y is.

solve for $\frac{dy}{dx}$

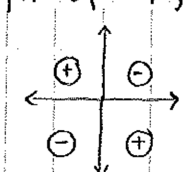
$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

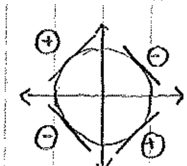
Check against reason:

	x	y	expected dy/dx	$-x/y$
✓	0	1	0	$-\frac{0}{1} = 0$
✓	0	-1	0	$-\frac{0}{-1} = 0$
✓	$\sqrt{2}/2$	$\sqrt{2}/2$	-1	$-\frac{\sqrt{2}}{2} \cdot \frac{2}{\sqrt{2}} = -1$
✓	$\sqrt{2}/2$	$-\sqrt{2}/2$	1	$-\frac{\sqrt{2}}{-2} \cdot \frac{2}{\sqrt{2}} = 1$

Sign of $-x/y$:

✓

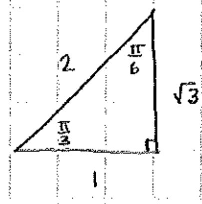




ex. Find all points on $x^2 + y^2 = 1$ where slope of tangent line is $\sqrt{3}$.

$$-\frac{x}{y} = \sqrt{3} \rightsquigarrow \text{Quadrants } 2 \,\&\, 4$$

$$\frac{y}{x} = -\frac{1}{\sqrt{3}} \rightsquigarrow \frac{\pi}{6}, \frac{\pi}{3}$$



$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\frac{\sin x}{\cos x} = -\frac{1}{\sqrt{3}}$$

$$\tan x = -\frac{1}{\sqrt{3}}$$

$$\left(-\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) \text{ or } \left(\cos\frac{\pi}{6}, -\sin\frac{\pi}{6}\right)$$

ex. Calculate $\frac{dy}{dx}$ if :

(1)
$$xy = 1$$
$$(1)(y) + (x)\left(\frac{dy}{dx}\right) = 0$$
$$(x)\left(\frac{dy}{dx}\right) = -y$$
$$\frac{dy}{dx} = -\frac{y}{x}$$

step 1: take $\frac{dy}{dx}$ of both sides
step 2: set them equal
step 3: solve for $\frac{dy}{dx}$

(2)
$$y + \sin x = xy$$
$$\frac{dy}{dx} + \cos x = y + (x)\left(\frac{dy}{dx}\right)$$
$$\frac{dy}{dx} - (x)\left(\frac{dy}{dx}\right) = y$$
$$\frac{dy}{dx}(1-x) = y$$
$$\frac{dy}{dx} = y/(1-x)$$

(3)
$$y/x = xy^4 + 1$$
$$\frac{x \cdot \frac{dy}{dx} - y}{x^2} = 0 + y^4 + 4xy^3 \cdot \frac{dy}{dx}$$
$$x\frac{dy}{dx} - y = x^2y^4 + 4x^3y^3 \cdot \frac{dy}{dx}$$
$$\frac{dy}{dx}\left(\frac{1}{x} - 4x^3y^3\right) = \frac{y}{x^2} + y^4$$
$$\frac{dy}{dx} = \frac{\frac{y}{x^2} + y^4}{\frac{1}{x} - 4xy^3}$$

step 2:
$$xy = y + \sin x$$
$$xy - y = \sin x$$
$$(x-1)y =$$
$$\frac{\sin x}{x-1}$$

Q. Where is the slope of the tangent line?

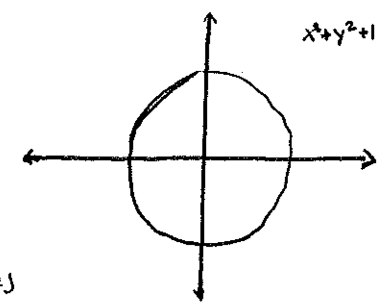
$$\frac{dy}{dx} = \frac{y - \cos x}{1-x} \; : x = 1$$

If curve is defined → vertical tangents, if not → undefined.

Warm Up:

1. $\frac{d}{dx}\left(f(x) * x\right) \rightarrow f(x) + x f'(x)$
2. $\frac{d}{dx}\left(f(x) + \sin(x)\right) \rightarrow f'(x) + \cos(x)$
3. $\frac{d}{dx}\left(f(x)/x\right) \rightarrow 2f(x)\,f'(x)$
4. $\frac{d}{dx}\left((f(x))^2\right) \rightarrow \frac{x f'(x) - f(x)}{x^2}$
5. $\frac{d}{dx}\left(1 + [x(f(x))^4]\right) \rightarrow$ look online
6. $\frac{d}{dx}\left(\cos(x e^{f(x)})\right) \rightarrow$

IMPLICIT FUNCTIONS



- might not be a function but a curve that can be expressed as $F(x,y)=c$.
- There are lots of things that can't be written or expressed as functions but can be expressed as implicit ones. ex: $\underline{x^2+y^2+1}$ (THE UNIT CIRCLE)

THUS far we have had to solve for y:

positive upper half: $y = \sqrt{1-x^2}$

lower negative half: $y = -\sqrt{1-x^2}$

ex: what is $\frac{dy}{dx}$ at $x = \frac{\sqrt{2}}{2}$? you need $y = \frac{\sqrt{2}}{2}$

$\rightarrow$ take $y = \sqrt{1-x^2}$ and differentiate

how about: $x = \frac{\sqrt{2}}{2}$, $y = \frac{\sqrt{2}}{2}$ must switch to another function.

INSTEAD: use implicit differentiation. Your answer must be a function of both x and Y.

STRATEGY: Think of y as a function of x. (back in warm up $y = f(x)$)

GOAL: given $F(x,y) = c$, calculate $\frac{dy}{dx}$ in terms of both x and y

what is $\frac{d}{dx}\left(x^2 + f^2(x)\right) \rightarrow$ chain rule $\rightarrow 2x + 2f(x)f'(x)$

what is $\frac{d}{dx} 1 = 0$

If I want to find $\frac{dy}{dx}$, where y and x are related by $x^2+y^2=1$, (*)

I start by taking $\frac{d}{dx}$ of both sides of (*)

$$\frac{d}{dx} 1 = 0 \rightarrow \frac{d}{dx}\left(x^2+y^2\right) \rightarrow 2x + 2y\frac{dy}{dx} \rightarrow \text{solve } 0 = 2x + 2y\frac{dy}{dx} \text{ for } \frac{dy}{dx}: \ \frac{dy}{dx} = \frac{-2x}{2y} \rightarrow \frac{-x}{y}$$

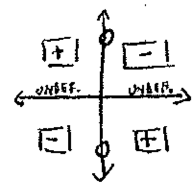
$$\begin{array}{l} 0 = 2x + 2y\frac{dy}{dx} \\ -2x \quad -2x \end{array} \rightarrow \frac{-2x}{2y} = \frac{2y}{2y}\frac{dy}{dx} \rightarrow \frac{-2x}{2y}$$

NEXT PAGE $\longrightarrow$

our answer of $\frac{-x}{y}$ can be checked against the unit ~~circle~~ circle:

x	y	what we expect of $\frac{dy}{dx}$	$-x/y$
0	1	0	$-0/1 = 0$ ✓
0	-1	0	$-0/-1 = 0$ ✓
$\sqrt{2}/2$	$\frac{\sqrt{2}}{2}$	-1	$-(\frac{\sqrt{2}}{2})/(\frac{\sqrt{2}}{2}) = -1$ ✓
$\sqrt{}$	$-\frac{\sqrt{2}}{2}$	1	$-(\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) = 1$ ✓

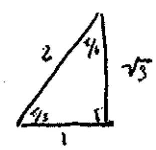
Sign of $-x/y$



ex: Find all points on ~~[illegible]~~ $x^2 + y^2 = 1$ where slope of the tangent is $\sqrt{3}$

solve: $-\frac{x}{y} = \sqrt{3} \rightarrow \frac{y}{x} = -\frac{1}{\sqrt{3}} \rightarrow$ (find angles where $-\frac{1}{\sqrt{3}} = \frac{y}{x} = \frac{\sin(x)}{\cos(x)} = \tan(x)$)

IDENTITY TREAMOUR ⟶



$\rightarrow \tan(\pi/6) = \frac{1}{\sqrt{3}} \rightarrow$ $x = -\cos(\pi/6)$ or $\cos(\pi/6)$
$y = \sin(\pi/6)$ or $-\sin(\pi/6)$

MORE EXAMPLES:

Calculate $\frac{dy}{dx}$ if ....

1. $xy = 1 \rightarrow x\frac{dy}{dx} + y \rightarrow \frac{d}{dx}(1) = 0 \rightarrow x\frac{dy}{dx} + y = 0 \rightarrow \boxed{\frac{dy}{dx} = \frac{-y}{x}}$ (both sides)

(you can factor out $\frac{dy}{dx}$ here)

2. $y + \sin(x) = xy \rightarrow \frac{d}{dx}(y + \sin(x)) = \frac{dy}{dx} + \cos(x) = \frac{d}{dx}(xy) = x\frac{dy}{dx} + y \rightarrow \frac{dy}{dx} - x\frac{dy}{dx} = y - \cos(x)$ $\frac{dy}{dx}(1-x) = \boxed{\frac{dy}{dx} = \frac{(y - \cos(x))}{(1-x)}}$
3. $y/x = xy^4 + 1 \rightarrow \frac{y'x - y}{x^2} = (y^4 + 4xy^3y') \rightarrow \frac{y'}{x} - \frac{y}{x^2} = y^4 + 4xy^3y' \rightarrow y'/x - 4xy^3y' = \frac{y}{x^2} + y^4 \rightarrow$ factor $\rightarrow y'(1/x - 4xy^3) = \frac{y}{x^2} + y^4$
4. $\cos(xe^y) = y^2$

$$\boxed{so: y' = \frac{\frac{y}{x^2} + y^4}{(\frac{1}{x} - 4xy^3)}}$$

> STEPS: 1. Take $\frac{d}{dx}$ of both sides
> 2. set them equal
> 3. Solve for $\frac{dy}{dx}$

Q: If $y + \sin(x) = xy$ where is the slope of the tangent line undefined? (at vertical)

go to $\frac{dy}{dx}$ and ask "where is this undefined?"

set from #2 $\frac{dy}{dx} = \frac{y - \cos(x)}{1-x}$: $\underline{x = 1}$

if the curve is defined at $x=1$ it means we have a vertical tangent, and if not it is undefined.

go to original function: $xy = y + \sin(x) \rightarrow xy - y = \sin(x)$

$(x-1)y = \sin(x)$

$y = \frac{\sin(x)}{x-1}$

This is undefined at $x=1$. The tangent is undefined at $x=1 \rightarrow$ means we have a vertical asymptote.

> with implicit functions you can have vertical tangents, if $\frac{dy}{dx}$ is undefined but y is defined

1/26 Warm-Up

Suppose $f(x)$ is a differentiable function. Then what are (in terms of x and $f(x)$, $f'(x)$)

1. $\frac{d}{dx}(f(x)\cdot x) = f(x)\cdot 1 + f'(x)(x) = \underline{f(x) + f'(x)x}$
2. $\frac{d}{dx}(f(x) + \sin(x)) = \underline{f'(x) + \cos x}$
3. $\frac{d}{dx}\left(\frac{f(x)}{x}\right) = \underline{\frac{f'(x)\cdot x - f(x)}{x^2}}$
4. $\frac{d}{dx}(f(x))^2 = \underline{2(f(x))\cdot f'(x)}$
5. $\frac{d}{dx}\, 1 + [x\cdot(f(x))^4] = \frac{d}{dx}\, 1 + x\cdot(f(x))^4 = \underline{0 + f^4(x) + 4x f(x)^3 f'(x)}$
6. $\frac{d}{dx}(\cos(x\cdot e^{f(x)})) = \underline{-\sin(xe^{f(x)})\cdot(e^{f(x)} + x f'(x) e^{f(x)})}$

* Implicit function: might not be a function, but it's a curve which can be expressed as $F(x,y) = C$

ex: unit circle

$x^2 + y^2 = 1$

... so far, have to solve for y

* $y = \sqrt{1-x^2}$ or $y = -\sqrt{1-x^2}$

ex) What is $\frac{dy}{dx}$ @ $x = \frac{\sqrt{2}}{2}$? need $y = \frac{\sqrt{2}}{2}$

$\Rightarrow$ take $y = \sqrt{1-x^2}$ and differentiate.

- How about @ $x = \sqrt{2}/2$, $y = -\frac{\sqrt{2}}{2}$ * Now must switch to another function!

Instead: Use "implicit differentiation"

- Goal: given $F(x,y) = C$ calculate $\frac{dy}{dx}$ in terms of both x and y

* Strategy: think of y as a function of x!

(back to warm up think $y = f(x)$)

If I asked what is $\frac{d}{dx}(x^2+f^2(x))$?

$= 2x + 2f(x)\cdot f'(x)$

What is $\frac{d}{dx}1 = 0$

If I want to find $\frac{dy}{dx}$ where y and x are related by $x^2+y^2=1$, (*)

I start by taking $\frac{d}{dx}$ of both sides of (*)

$$\frac{d}{dx}1 = 0 = \frac{d}{dx}(x^2+y^2)$$

$$2x + 2\cdot y\cdot \boxed{\frac{dy}{dx}}$$

Solve! $0 = 2x + 2y\cdot\frac{dy}{dx}$

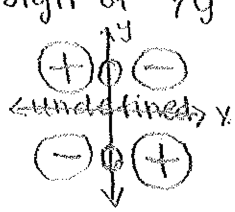
$-2x = 2y\frac{dy}{dx}$

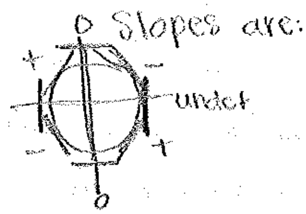
$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

Check It!

x	y	expect $\frac{dy}{dx}$	$-x/y$
0	1	0	0 ✓
0	-1	0	0 ✓
$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	1 ✓
$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	-1 ✓

Sign of $-x/y$

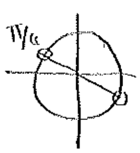




Ex Find all points on $x^2+y^2=1$ where slope of the tangent line is $\sqrt{3}$.

$$-\frac{x}{y} = \sqrt{3}$$

$\hookrightarrow \frac{y}{x} = -\frac{1}{\sqrt{3}}$ find angles where $-\frac{1}{\sqrt{3}} = \frac{y}{x} = \frac{\sin(x)}{\cos(x)} = \tan(x)$



$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$$\left\{\begin{matrix} x = -\cos(\pi/6) & \text{or} & \cos(\pi/6) \\ y = \sin(\pi/6) & & -\sin(\pi/6) \end{matrix}\right\}$$

Calculate $\frac{dy}{dx}$ if ...

① $xy = 1$

$\frac{d}{dx} xy = x\frac{dy}{dx} + y = 0 = \frac{d}{dx}1$

$\boxed{\frac{dy}{dx} = -y/x}$

② $y + \sin(x) = xy$

$\frac{d}{dx}\, y + \sin(x) = \frac{dy}{dx} + \cos x = x\frac{dy}{dx} + y = \frac{d}{dx}x$

$\frac{dy}{dx} - x\frac{dy}{dx} = y - \cos x$

$\frac{dy}{dx}(1-x) =$

$\boxed{\frac{dy}{dx} = \frac{y - \cos x}{1-x}}$

③ $\frac{y}{x} = xy^4 + 1$

$\frac{d}{dx}\frac{y}{x} = \frac{\frac{dy}{dx}x - y}{x^2} = y^4 + 4xy^3\frac{dy}{dx} = \frac{d}{dx}xy^4 + 1$

$\frac{dy}{dx}\cdot\frac{1}{x} - 4xy^3\frac{dy}{dx} = y^4 + \frac{y}{x^2}$

$\frac{dy}{dx}\left(\frac{1}{x} - 4xy^3\right) = y^4 + \frac{y}{x^2}$

$\boxed{\frac{dy}{dx} = \frac{y^4 + \frac{y}{x^2}}{\frac{1}{x} - 4xy^3}}$

④ $\cos(xe^y) = y^2$

(didn't finish)

Q: If $y + \sin(x) = xy$ Where is slope of tangent line undefined? (or vertical)

Go to $\frac{dy}{dx}$ and ask where that is undefined!

$\frac{dy}{dx} = \frac{y - \cos x}{1-x}$: $x = 1$

If the curve is def. @ $x = 1 \Rightarrow$ vert tangent

if not $\Rightarrow$ undefined

Step 2: $xy = y + \sin x$

$xy - y = \sin x$

$y(x-1) = \sin x$

$y = \frac{\sin x}{x-1}$

*Undefined at $x = 1$

$\Rightarrow$ Vertical asymptote

*MORAL: With implicit functions you can have vertical tangents... If $\frac{dy}{dx}$ is undef, but y is defined