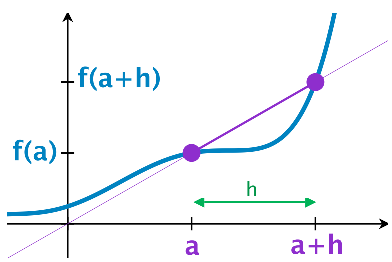
Worksheet – The tangent line problem

We've been building towards studying rates of change, e.g.

rate at which position changes versus time (= velocity);
rate at which birthrate changes versus average household income;
rate at which profit margin changes versus production volume.

In general, the instantaneous rate of change of a function $f(x)$ versus x at a point a is given by the limit of the *difference quotient*:

$$\text{inst. rate of change} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$



Another word for the instantaneous rate of change of a function $f(x)$ at a point a is the **derivative** of $f(x)$ at $x = a$, written $f'(a)$. So

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.}$$

The derivative also has a *geometric* interpretation:

$$\boxed{f'(a) = \text{slope of the line tangent to } y = f(x) \text{ at } x = a.}$$

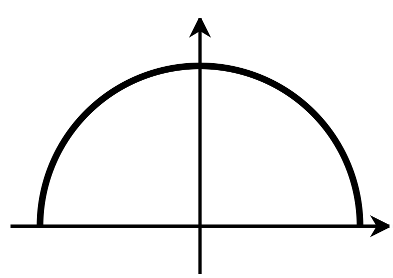
Example 1: Below is a graph of the function $f(x) = \sqrt{1 - x^2}$ (the half circle with radius 1). Without calculating any limits, what is

(a) $f'(0)$?

(b) $f'(\frac{\sqrt{2}}{2})$?

(c) $f'(-\frac{\sqrt{2}}{2})$?

[hint: for (b) and (c), draw a line from the origin to the point in question. What angle does that make with the x-axis? What is the slope of that line? For a circle, the line tangent at a point is perpendicular to the ray from the center to the point.]



Once you have the slope, it's pretty easy to write down the equations for the tangent line using point-slope form:

$$y = m(x - x_0) + y_0 \qquad \text{becomes} \qquad \boxed{y = f'(a)(x - a) + f(a).}$$

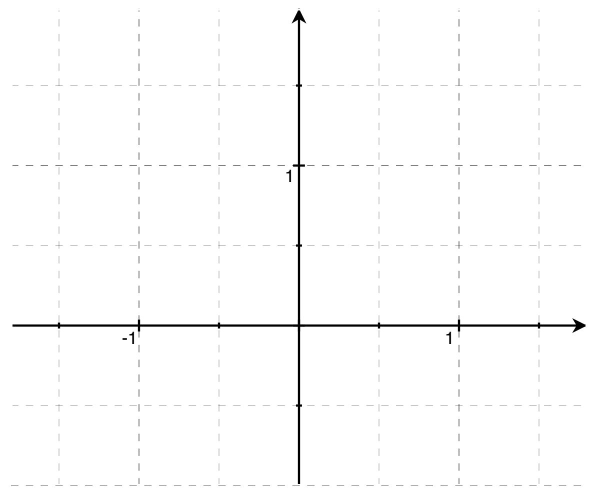
Example 2: What is the equation for the line tangent to $f(x) = \sqrt{1 - x^2}$ at

(a) $x = 0$?

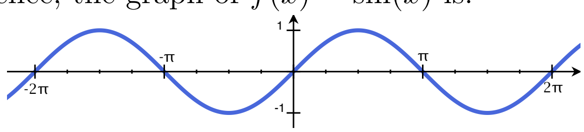
(b) $x = \frac{\sqrt{2}}{2}$?

(c) $x = -\frac{\sqrt{2}}{2}$?

Check your answers by first sketching the lines you wrote down in (a)-(c), and *then* sketching the function $f(x) = \sqrt{1 - x^2}$ on the axes to the right.



Example 3: For reference, the graph of $f(x) = \sin(x)$ is:



(a) The function $\sin(x)$ has infinitely many points $x = a$ where $f'(a) = 0$. What are they?

(b) There are exactly two horizontal lines which are tangent to $\sin(x)$. What are they?

(c) [Bonus] Can you think of a function which has infinitely many points where $f'(a) = 0$, not just anywhere, but between $x = 0$ and $x = \pi$? [hint: think back to the day we did limits. There is some function $g(x)$ which we could plug into $\sin(x)$ which will make $\sin(g(x))$ a good answer to this question.]

Answers: $1(a) : 0$, $(b) : -1$, $(c) : 1$, $2(a) : y = 1$, $(b) : y = -x + \sqrt{2}$, $(c) : x + \sqrt{2}$, $3(a) : \frac{\pi}{2} + \pi k$, $(b) : y = \pm 1$.

Calculating derivative using limits

Recall from last Friday that we have a few tricks for calculating limits $\lim_{x \to a} g(x)$:

1. **Plugging in:** If $g(x)$ is continuous, and $g(a)$ is defined, then $\lim_{x \to a} g(x) = g(a)$.

   For example, $\lim_{x \to 2} \frac{x+1}{x-3} = \boxed{}$

2. **Factor and cancel:** If $g(x)$ is rational, and $g(a)$ is **not** defined, but a is a root of the numerator and denominator, then factor and cancel:

   For example,

   $$\lim_{x \to 2} \frac{x+1}{x-2} \quad \text{is } \textbf{undefined},$$

   but

   $$\lim_{x \to 2} \frac{x-2}{x^2-4} = \qquad = \boxed{}$$

3. **Expand and cancel:** It's like spring cleaning – make a mess, and then clean up!.

   For example, since $(x+2)^3 = x^3 + 6x^2 + 12x + 8$,

   $$\lim_{x \to 0} \frac{x}{(x+2)^3 - 8} = \qquad = \boxed{}$$

4. **Common denominators:** If you have a sum or difference of fractions, find a common denominator and see what happens.

   For example,

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x(x+1)}\right)= \qquad\qquad = \boxed{}$$

5. **Multiply top and bottom by the conjugate:** If you have a difference of square roots (like $\sqrt{a}-\sqrt{b}$), you can multiply and divide by the *conjugate*, $\sqrt{a}+\sqrt{b}$. This is useful because

$$\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)=\left(\sqrt{a}\right)^2-(\sqrt{b})^2=\boxed{a-b}$$

   For example, try multiplying by $\dfrac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$: (notice $2=\sqrt{4}$)

$$\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2}= \qquad\qquad = \boxed{}$$

Answers: $1:-3,\ \ 2:\frac{1}{4},\ \ 3:\frac{1}{12},\ \ 4:1,\ \ 5:4.$

Now that we have these tools, let's calculate some derivatives!

(A) Use the limit definitions the derivative of $f(x) = x^2$ at $x = 1$:

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{(1+h)^2 - (1)^2}{h} \\ &= \end{aligned}$$

$$= \boxed{}$$

(B) Use the limit definitions the derivative of $f(x) = x^3$ at $x = -2$:

$$\begin{aligned} f'(-2) &= \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h\to 0} \frac{(-2+h)^3 - (-2)^3}{h} \\ &= \end{aligned}$$

careful! $(-2)^3 = -8$, so $-(-2)^3 = 8$

$$= \boxed{}$$

(C) Use the limit definitions the derivative of $f(x) = \frac{1}{x}$ at $x = 3$:

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$=$$

$$= \boxed{}$$

(D) Use the limit definitions the derivative of $f(x) = \sqrt{x}$ at $x = 5$:

$$f'(3) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$$

$$=$$

$$= \boxed{}$$

Answers: $A: 2, \quad B: 12, \quad C: -\frac{1}{9}, \quad D: \frac{1}{2\sqrt{5}}$

Back to tangent line equations:

Use your answers to A-D on the previous two pages to calculate the lines tangent to

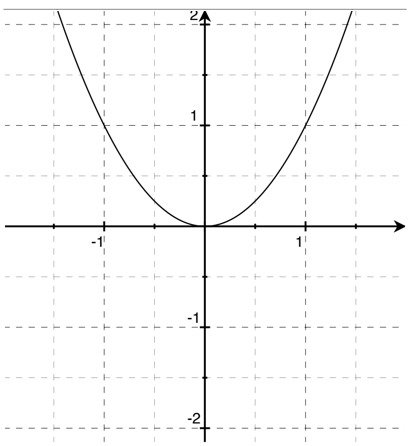
(a) $f(x) = x^2$ at $x = 1$

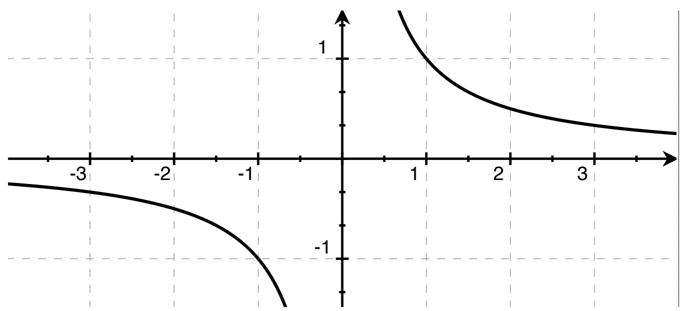
(b) $f(x) = x^3$ at $x = -2$

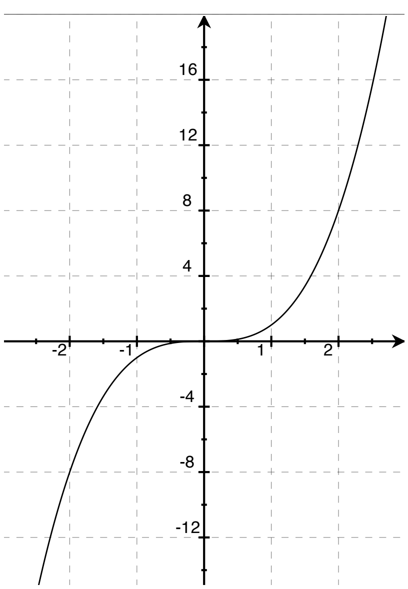
(c) $f(x) = \frac{1}{x}$ at $x = 3$

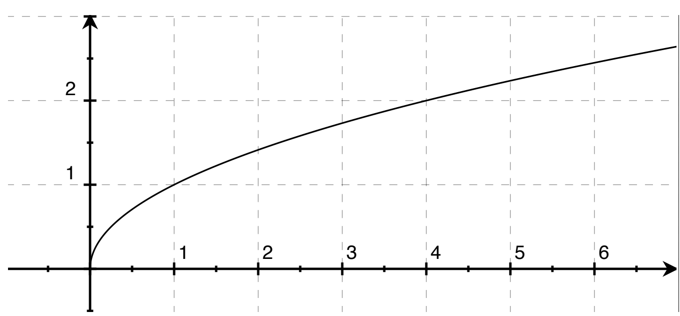
(d) $f(x) = \sqrt{x}$ at $x = 5$

Check your answers by sketching the lines from (a)-(d) on onto the appropriate graphs below:









Answers: $a : y = 2x - 1$, $b : y = 12x + 16$, $c : -\frac{1}{9}x + \frac{2}{3}$, $d : \frac{1}{2\sqrt{5}}x + \frac{\sqrt{5}}{2}$

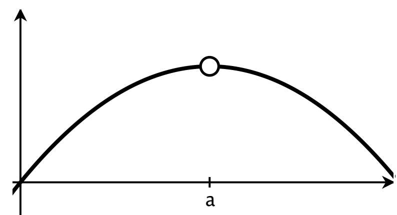
When can we take derivatives?

Not all functions have derivatives at all places. Before calculating $f'(a)$, first ask ...

1. **Is $f(x)$ defined at $x = a$?**

   For example, even if it looks like you could draw a tangent line, if there's a hole, $f'(a)$ **does not exist**!

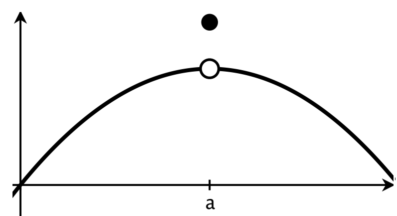
   (It's tempting to say $f'(a)$ exists here in part because $f(x)$ has a *continuous extension* at a.)



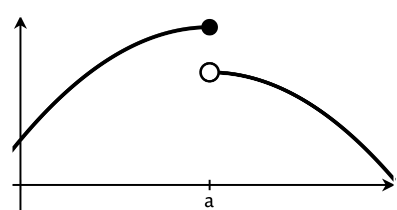
2. **Is $f(x)$ continuous at $x = a$?**

   For example, even if it looks like you could draw a tangent line, if there's a jump, $f'(a)$ **does not exist**!

   (Try drawing just one line that is tangent to that isolated point. It's tempting to say $f'(a)$ exists here in part because $f(x)$ has a *removable discontinuity* at a.)

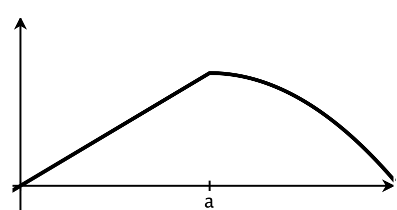


   Again, even if the slope looks the same from the left and from the right, if there's a discontinuity, $f'(a)$ **does not exist**!



3. **Is there a "corner" at $x = a$?**
   Next we'll explore how to find these algebraically, but if there's a sharp corner at $x = a$, then $f'(a)$ **does not exist**!

   (Try drawing just one line that is tangent to that corner)



What's wrong with corners?

Let $f(x) = \begin{cases} x^2 & x < 2, \\ x+2 & x \geq 2. \end{cases}$

(a) Verify that $f(x)$ is continuous at $x = 2$.

(b) Sketch a graph of $f(x)$.

(c) Estimate, and then calculate the *right sided derivative*.

(i) Estimate:

a	3	2.5	2.1	2
$f(a)$				

h	$f(2+h)-f(2)$	$\frac{f(2+h)-f(2)}{h}$
1		
1/2		
1/10		

(ii) Explain why $\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-} \frac{(2+2+h)-(2+2)}{h}$.

(iii) Calculate $\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}$.

(d) Estimate, and then calculate the *left sided derivative*. (OK to use a calculator for (i))

(i) Estimate:

a	1	1.5	1.9	2
$f(a)$				

h	$f(2+h)-f(2)$	$\frac{f(2+h)-f(2)}{h}$
-1		
$-1/2$		
$-1/10$		

(ii) Explain why $\lim_{h \to 0^-} \dfrac{f(2+h)-f(2)}{h} = \lim_{h \to 0^-} \dfrac{(2+h)^2-(2)^2}{h}$.

(iii) Calculate $\lim_{h \to 0^-} \dfrac{f(2+h)-f(2)}{h}$.

(e) Compare your answers to (b) and (c), and explain why $\lim_{h \to 0} \dfrac{f(2+h)-f(2)}{h}$ does not exist. Explain why $f'(2)$ does not exist.

(f) Sketch graphs of the following functions and identify points where each function is not differentiable:

$$f(x) = |x| \qquad g(x) = |x-2| \qquad h(x) = \big|4-|x-2|\big| \qquad \psi(x) = \frac{|x|}{x} \qquad \phi(x) = \begin{cases} x^2 & x < 0, \\ x^4 & x \geq 0. \end{cases}$$

[hint: for $h(x)$, start by plotting some points, and then find points where $x - 2$ goes from positive to negative, and where $4 - |x - 2|$ goes from positive to negative.]

Answers: a: check each requirement, c (iii): 1, d (iii): 4, e: do the two sides meet?

$f(x) : x = 0, \quad g(x) : x = 2, \quad h(x) : x = -2, 2, 6, \quad \psi(x) : x = 0, \quad \phi(x) : \text{no } x!$