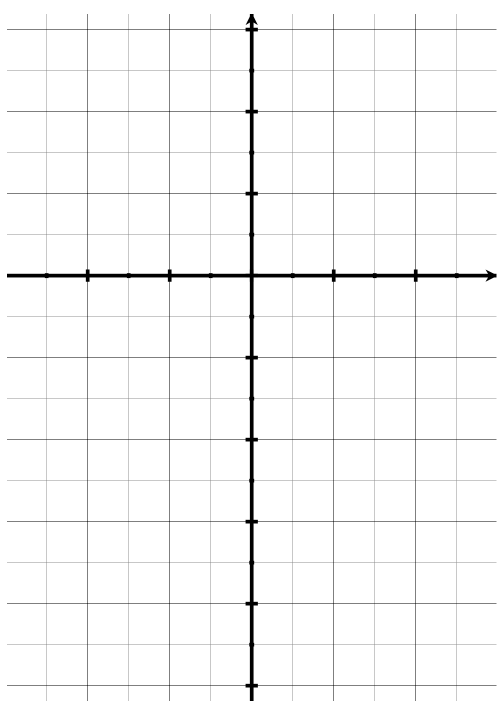
Slope Fields and Euler's Method

Warm up

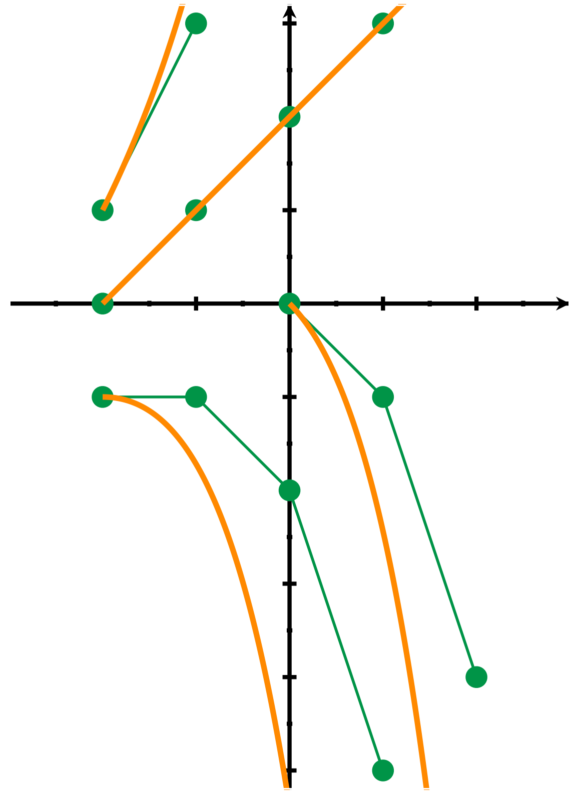
Suppose $\frac{dy}{dx} = y - x - 1$. Sketch part of the slope field for the following points:

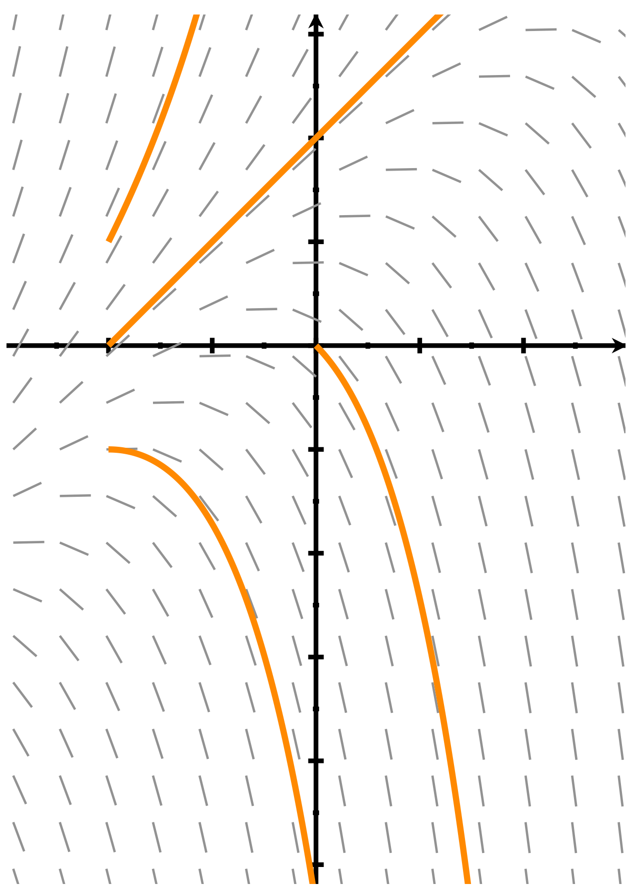
x	y	$\frac{dy}{dx}$
-2	0	
-2	1	
-2	-1	
-1	1	
-1	-1	
0	2	
0	0	
0	-2	
1	-1	



Suppose $\frac{dy}{dx} = y - x - 1$. Sketch part of the slope field for the following points:

x	y	$\frac{dy}{dx}$
-2	0	1
-2	1	2
-2	-1	0
-1	1	1
-1	-1	-1
0	2	1
0	0	-1
0	-2	-3
1	-1	-3





Euler's Method

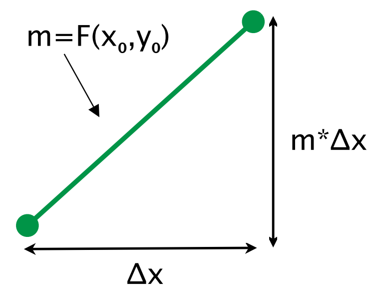
Assume that you have an IVP that looks like

$$\frac{dy}{dx} = F(x,y), \qquad y(x_0) = y_0$$

Pick an increment of x-steps, called Δx.

Start at (x_0, y_0), and plot a segment with run Δx and slope $F(x_0, y_0)$. The end is (x_1, y_1). From each (x_i, y_i), generate (x_{i+1}, y_{i+1}) by plotting a segment with run Δx and slope $F(x_i, y_i)$.

$$\begin{aligned} x_{i+1} &= x_i + \Delta x \\ y_{i+1} &= y_i + \Delta x * F(x_i, y_i) \end{aligned}$$

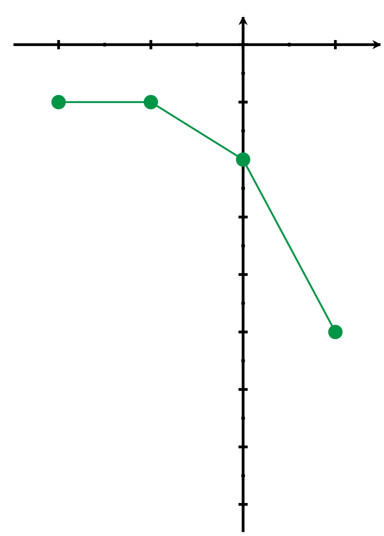


Back to $\frac{dy}{dx} = y - x - 1 = F(x,y), \quad P_0 = (-2,-1).$

$$x_{i+1} = x_i + \Delta x$$
$$y_{i+1} = y_i + \Delta x * (y_i - x_i - 1)$$

Example: $\Delta x = 1$

i	x_i	y_i	$F(x_i, y_i)$	y_{i+1}
0	-2	-1	0	-1+1*0
1	-1	-1	-1	-1+1*(-1)
2	0	-2	-3	-2+1*(-3)
3	1	-5		



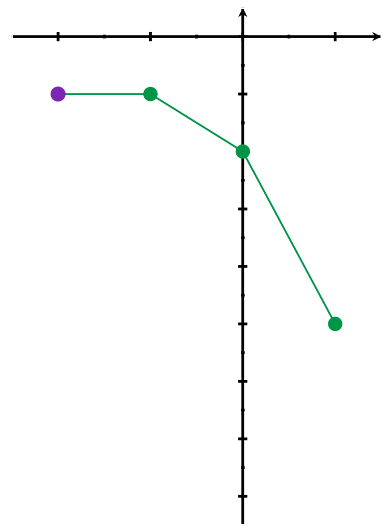
$\Delta x = 1, y_3 = -5$

Back to $\frac{dy}{dx} = y - x - 1 = F(x, y), \quad P_0 = (-2, -1).$

$$x_{i+1} = x_i + \Delta x$$
$$y_{i+1} = y_i + \Delta x * (y_i - x_i - 1)$$

Example: $\Delta x = 1/2$

i	x_i	y_i	$F(x_i, y_i)$	y_{i+1}
0	-2	-1		
1				
2				
3				



$\Delta x = 1, y_3 = -5$

Back to $\frac{dy}{dx} = y - x - 1 = F(x, y), \quad P_0 = (-2, -1)$.

$$x_{i+1} = x_i + \Delta x$$
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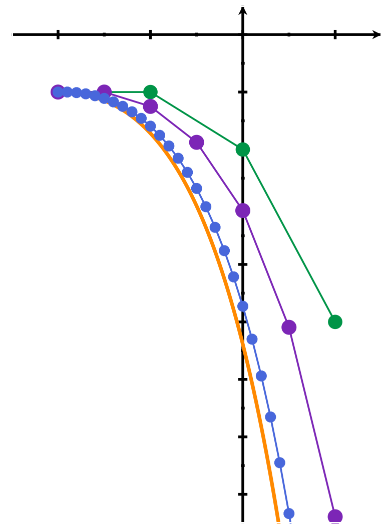


$\Delta x = 1, y_3 = -5$

$\Delta x = .1, y_{30} = -14.449$

Example: $\Delta x = 1/2$

i	x_i	y_i	$F(x_i, y_i)$	y_{i+1}
0	-2	-1	0	-1+$\frac{1}{2}$*0
1	-$\frac{3}{2}$	-1	-$\frac{1}{2}$	-1+$\frac{1}{2}$*(-$\frac{1}{2}$)
2	-1	-$\frac{5}{4}$	-$\frac{5}{4}$	-$\frac{5}{4}$+$\frac{1}{2}$*(-$\frac{5}{4}$)
3	-$\frac{1}{2}$	-$\frac{15}{8}$	-$\frac{19}{8}$	-$\frac{15}{8}$+$\frac{1}{2}$*(-$\frac{19}{8}$)
4	0	-3.0625	-4.0625	
5	$\frac{1}{2}$	-5.0938	-6.5938	
6	1	-8.3906 $= y_6$		

Actual solution: $y = -e^{x+2} + x + 2$, $\qquad y(1) \approx -17.0855$

Another example

If $\dfrac{dy}{dx} = \dfrac{y^2 + 3y}{x + 4}$ and $y(0) = -1$, what is $y(3)$?

Estimate: Try $\Delta x = 1$
$x_0 = 1$, $y_0 = -1$

$m_0 = \frac{(-1)^2 + 3(-1)}{0+4} = -0.5$ $\quad x_1 = 0 + 1 = 1 \quad y_1 = -1 + 1 * (-0.5) = -1.5$

$m_1 =$ $\quad x_2 =$ $\quad y_2 =$

$m_2 =$ $\quad x_3 =$

$y_3 =$

Spreadsheet set up:

in cell...	A1	B1	C1	D1	A2	B2	C2	F2
put...	i	xi	yi	mi	0	x_0	y_0	Δx

(In the first example, x_0 was -2 and y_0 was -1 and Δx was 1, $\frac{1}{2}$,...)

in cell...	D2	B3	C3
put...	$=F(B2, C2)$	=B2+$F$2	=C2+$F$2*D2
		$x_{i+1} = x_i + \Delta x$	$y_{i+1} = y_i + \Delta x * m_i$

(In the last example, $F(B2, C2)$ was C2−B2−1)

i	xi	yi	mi
0	0	-1	-0.5
1	1	-1.5	-0.45
2	2	-1.95	-0.34125
3	3	-2.29125	

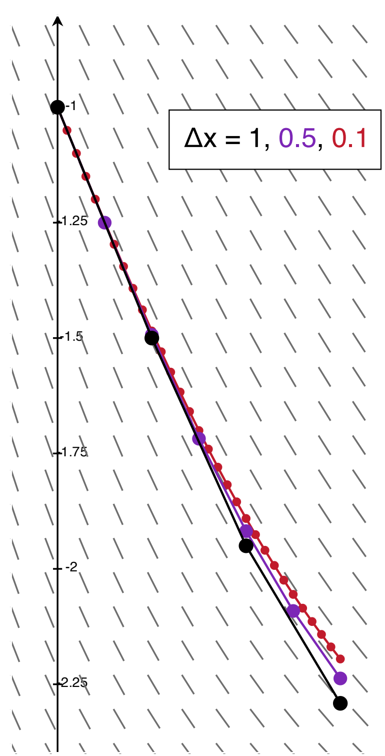
Dx
1

i	xi	yi	mi
0	0	-1	-0.5
1	0.5	-1.25	-0.486111111
2	1	-1.493055556	-0.449990355
3	1.5	-1.718050733	-0.40044616
4	2	-1.918273813	-0.34584117
5	2.5	-2.091194398	-0.292382951
6	3	-2.237385873	

Dx
0.5

i	xi	yi	mi
0	0	-1	-0.5
1	0.1	-1.05	-0.499390244
2	0.2	-1.099939024	-0.497607432
3	0.3	-1.149699768	-0.494718546
4	0.4	-1.199171622	-0.490795974
5	0.5	-1.24825122	-0.485916123
6	0.6	-1.296842832	-0.480158079
7	0.7	-1.34485864	-0.473602374
8	0.8	-1.392218877	-0.466329839
9	0.9	-1.438851861	-0.458420593
10	1	-1.48469392	-0.449953145
11	1.1	-1.529689235	-0.441003637
12	1.2	-1.573789599	-0.431645211
13	1.3	-1.61695412	-0.421947497
14	1.4	-1.659148869	-0.411976229
15	1.5	-1.700346492	-0.401792961
16	1.6	-1.740525788	-0.391454883
17	1.7	-1.779671277	-0.381014733
18	1.8	-1.81777275	-0.370520772
19	1.9	-1.854824827	-0.360016838
20	2	-1.890826511	-0.34954244
21	2.1	-1.925780755	-0.33913291
22	2.2	-1.959694046	-0.328819578
23	2.3	-1.992576004	-0.318629981
24	2.4	-2.024439002	-0.308588083
25	2.5	-2.05529781	-0.298714514
26	2.6	-2.085169262	-0.289026808
27	2.7	-2.114071942	-0.279539649
28	2.8	-2.142025907	-0.270265108
29	2.9	-2.169052418	-0.261212879
30	3	-2.195173706	

Dx
0.1



Back to "If $\frac{dy}{dx} = \frac{y^2+3y}{x+4}$ and $y(0) = -1$, what is $y(3)$?"

Can I just solve? Separate...

$$\int \frac{1}{y^2+3y}\,dy = \int \frac{1}{x+4}\,dx = \ln|x+4| + C$$

$$\frac{1}{3}\int \frac{1}{y}\,dy - \frac{1}{3}\int \frac{1}{y+3}\,dy =$$

check LHS: $\frac{1}{3}\left(\frac{1}{y} - \frac{1}{y+3}\right) = \frac{1}{3}\left(\frac{y+3-y}{y(y+3)}\right) = \frac{1}{3}\frac{3}{y^2+3y}$ ✓

$$\frac{1}{3}\ln|y| - \frac{1}{3}\ln|y+3| =$$

$$\frac{1}{3}\ln\left|\frac{y}{y+3}\right| =$$

Solve for y:

$$\frac{1}{3}\ln\left|\frac{y}{y+3}\right| = \ln|x+4| + C \quad \text{so} \quad \ln\left|\frac{y}{y+3}\right| = 3\ln|x+4| + 3C$$

Plug in point:

$$y(0) = -1 \qquad \text{means} \qquad \ln\left|\frac{-1}{-1+3}\right| = 3\ln|0+4| + 3C$$

$$\text{so } 3C = \underbrace{\ln(1/2) - 3\ln(4) = \ln\left(\frac{2^{-1}}{(2^2)^3}\right) = \ln(2^{-1-6})}_{\text{(computation one does in the privacy of one's home)}} = \ln(2^{-7})$$

Back to "If $\dfrac{dy}{dx} = \dfrac{y^2+3y}{x+4}$ and $y(0) = -1$, what is $y(3)$?"

Solve for y:

$$\frac{1}{3}\ln\left|\frac{y}{y+3}\right| = \ln|x+4| + C \quad \text{so} \quad \ln\left|\frac{y}{y+3}\right| = 3\ln|x+4| + 3C$$

$$= 3\ln|x+4| + \ln(2^{-7})$$

So

$$\left|\frac{y}{y+3}\right| = e^{3\ln|x+4| + \ln(2^{-7})} = e^{3\ln|x+4|}e^{\ln(2^{-7})} = \left(e^{\ln|x+4|}\right)^3 2^{-7}$$

$$= |x+4|^3 2^{-7}. \quad \text{So} \quad \frac{y}{y+3} = -\,2^{-7}|x+4|^3 \qquad (-\text{ b/c } y(0) = -1)$$

Therefore, if $A(x) = 2^{-7}|x+4|^3$, multiplying both sides by $y+3$ gives:

$$y = -A(x)(y+3) = -A(x) * y - A(x) * 3,$$

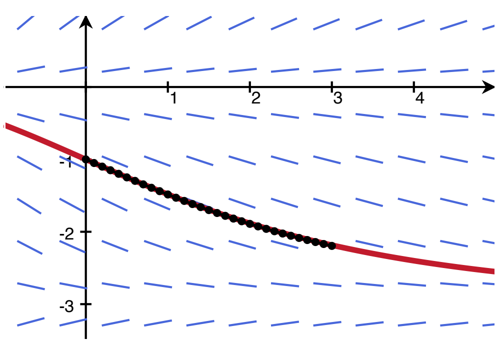
And so

$$-3A(x) = y + A(x) * y = (1 + A(x))y$$

So, finally, divide by $(1 + A(x))$ and sub back:

$$y = \frac{-3 * A(x)}{1 + A(x)} = \frac{-3 * 2^{-7}|x+4|^3}{1 + 2^{-7}|x+4|^3}$$

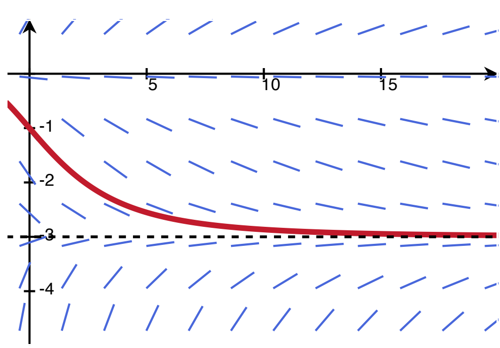
Compare $y = \frac{-3*2^{-7}|x+4|^3}{1+2^{-7}|x+4|^3}$ to our estimate:



$y(3) = \frac{-3*2^{-7}7^3}{1+2^{-7}7^3} = -2.18471\ldots$

Euler w $\Delta x = .1$: $y(3) \approx -2.19517\ldots$

Notice that $\lim_{x\to\infty} \frac{-3*2^{-7}|x+4|^3}{1+2^{-7}|x+4|^3} = \lim_{x\to\infty} \frac{-3*2^{-7}x^3}{2^{-7}x^3} = -3$



Extra practice

1. Perform the following approximations (by hand):
   (a) If $\frac{dy}{dx} = x + 2y$ and $y(0) = 1$, approximate $y(2)$ using $\Delta x = 1/2$.
   (b) If $\frac{dy}{dx} = xy$ and $y(1) = -1$, approximate $y(2)$ using $\Delta x = 1/2$.
   (c) If $\frac{dy}{dx} = y/x$ and $y(-2) = 1$, approximate $y(-1)$ using $\Delta x = 1/3$.
2. Why can't I approximate a solution to $\frac{dy}{dx} = y/x$ at $y(1)$ with the information that $y(-2) = 1$ using *any* Δx?
3. Which of the three problems in part 1 can you solve exactly? Compare to your approximation.