Exponential growth and decay

Warm-up

1. If $(A+B)x - 2A = 3x + 1$ for all x, what are A and B?
   (Hint: if it's true for *all* x, then the coefficients have to match up, i.e. $A + B = 3$ and $-2A = 1$.)
2. Find numbers (maybe not integers) A and B which satisfy
   $$\frac{3x+1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}.$$
   (Hint: Combine the fractions on the right side so that the denominators match on both sides. Then compare the numerators, and see problem 1.)
3. Find numbers (maybe not integers) A and B which satisfy
   $$\frac{1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}.$$
   (Hint: This is like problem 2 again, but when you set up your two equations for A and B, think of 1 like $0 * x + 1$.)
4. Use the last problem to calculate $\displaystyle\int \frac{1}{x^2-4}dx$.

Example 1: Suppose a bacteria culture grows at a rate proportional to the number of cells present. If the culture contains 700 cells initially and 900 after 12 hours, how many will be present after 24 hours?

The plan:

1. Put it into math, i.e. Write down an initial value problem.

1'. Look at slope fields to make sure the IVP makes sense.

2. Find the general solution to the IVP.
3. Plug in the points and find the particular solution.
4. Calculate the value of the solution when $t = 24$.

Step 1: Put into math. Initial value problem:

$$\frac{dy}{dt} = ky, \qquad y(0) = 700, \quad y(12) = 900$$

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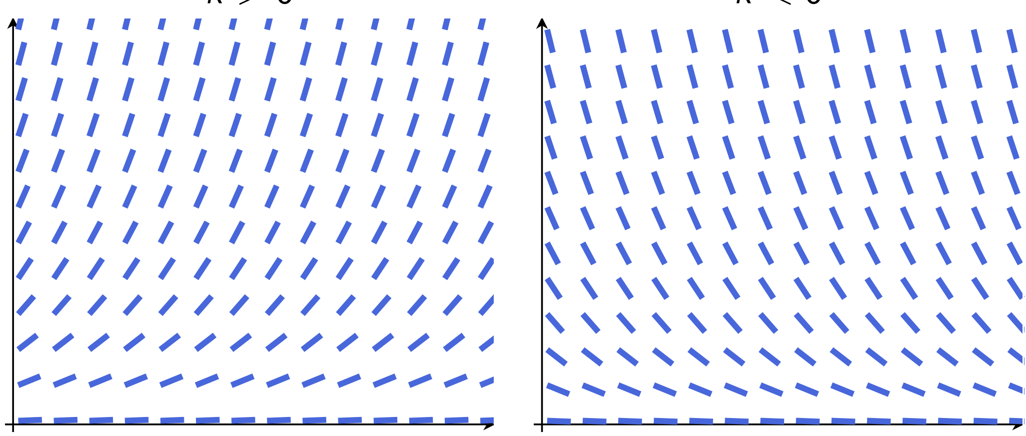
$k > 0$ $\qquad$ $k < 0$



$$\boxed{\frac{dy}{dt} = ky, \qquad y(0) = 700, \quad y(12) = 900}$$

Step 2: Find the general solution. To solve: Separate!

$$\int \frac{1}{y} dy = \int k \ dt$$

LHS: $\int \frac{1}{y} \ dy = \ln|y| + c_1$
RHS: $\int k \ dt = kt + c_2$
Putting it together:

$$\begin{aligned} \ln|y| = kt + C &\implies |y| = e^{kt+C} = e^C * e^{kt} \\ &\implies y = \pm e^C * e^{kt} = Ae^{kt}. \end{aligned}$$

General solution: $\boxed{y = Ae^{kt}}$

$$\boxed{\frac{dy}{dt} = ky, \qquad y(0) = 700, \quad y(12) = 900}$$

General solution: $\boxed{y = Ae^{kt}}$

Step 3: Plug in points and find particular solution

$$700 = y(0) = Ae^0 = A, \quad \text{so } y = 700e^{kt}$$

$$\begin{aligned} 900 = 700e^{12k} &\implies 12k = \ln(900/700) = ln(9/7) \\ &\implies k = \frac{1}{12}\ln(9/7) \approx \boxed{0.021} \end{aligned}$$

Particular solution: $\boxed{y = 700e^{t*\frac{1}{12}\ln(9/7)}}$

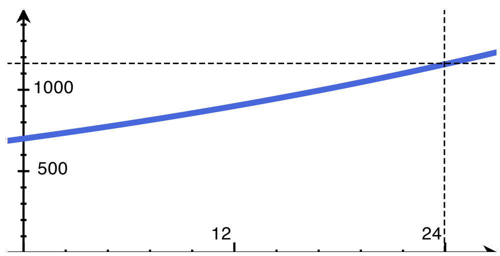
Note: another way to write this is

$$y = 700e^{t*\frac{1}{12}\ln(9/7)} = 700\left(e^{ln(9/7)}\right)^{t/12} = 700\left(\frac{9}{7}\right)^{t/12}$$

Example 1: Suppose a bacteria culture grows at a rate proportional to the number of cells present. If the culture contains 700 cells initially and 900 after 12 hours, how many will be present after 24 hours?

General solution: $\boxed{y = Ae^{kt}}$

Particular solution: $\boxed{y = 700\left(\frac{9}{7}\right)^{t/12}}$



$$y(24) = 700\left(\frac{9}{7}\right)^{24/12} = 700\left(\frac{9}{7}\right)^{2} = 8100/7 \approx 1157.14$$

Example 2: Objects heat or cool at a rate proportional to the difference between their temperature and the ambient temperature. Suppose a pie is pulled out of the oven (heated to 370°F), and put into a room that's 70°F. After 10 minutes, the center of the pie is 340°F.

(a) How hot is the pie after 20 minutes?

(b) How long will it take for the center of the pie to cool to 100°F?

The plan:

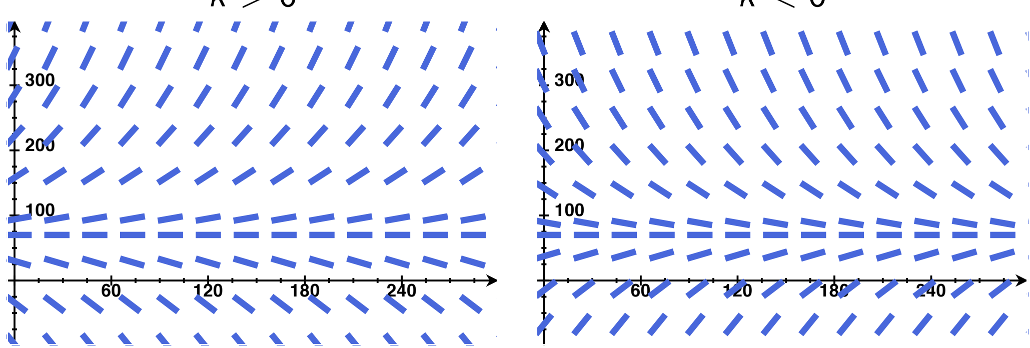
1. Put it into math, i.e. Write down an initial value problem.
2. Find the general solution to the IVP.
3. Plug in the points and find the particular solution.
4. Calculate the value of the solution when $t = 20$.
5. Solve for t when the solution is equal to 100.

Step 1: Put into math. Initial value problem:

$$\frac{dy}{dt} = k(y - 70), \qquad y(0) = 370, \quad y(10) = 340$$

IVP: $\frac{dy}{dt} = k(y - 70), \qquad y(0) = 370, \quad y(10) = 340$

$k > 0$ $\qquad$ $k < 0$



$$\text{IVP: } \frac{dy}{dt} = k(y-70), \qquad y(0) = 370, \quad y(10) = 340$$

Step 2: Find the general solution. To solve: Separate!

$$\int \frac{1}{y-70} dy = \int k \ dt$$

LHS: $\int \frac{1}{y-70} dy = \ln|y-70| + c_1$,

RHS: $\int kdt = kt + c_2$

Putting it together: $\ln|y-70| = kt + c$ (where $c = c_2 - c_1$).
So

$$y - 70 = \pm e^{kt+c} = \pm e^c * e^{kt} = Ae^{kt} \qquad \text{where } A = \pm e^c,$$

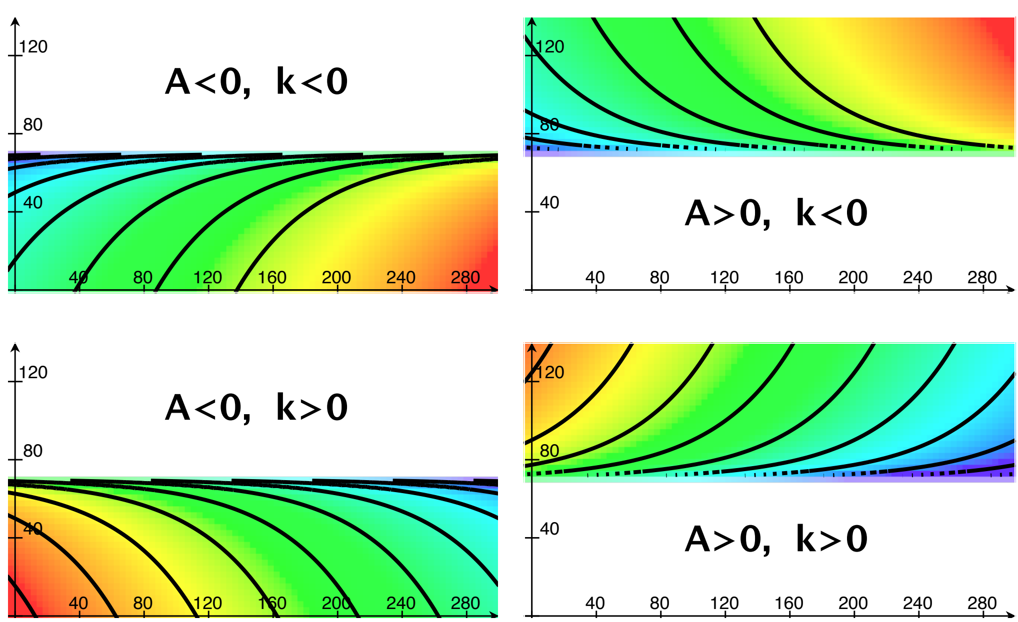
and so

$$\boxed{y = Ae^{kt} + 70}$$

$$\text{IVP: } \frac{dy}{dt} = k(y-70), \qquad y(0) = 370, \quad y(10) = 340$$

General solution: $\boxed{y = Ae^{kt} + 70}$

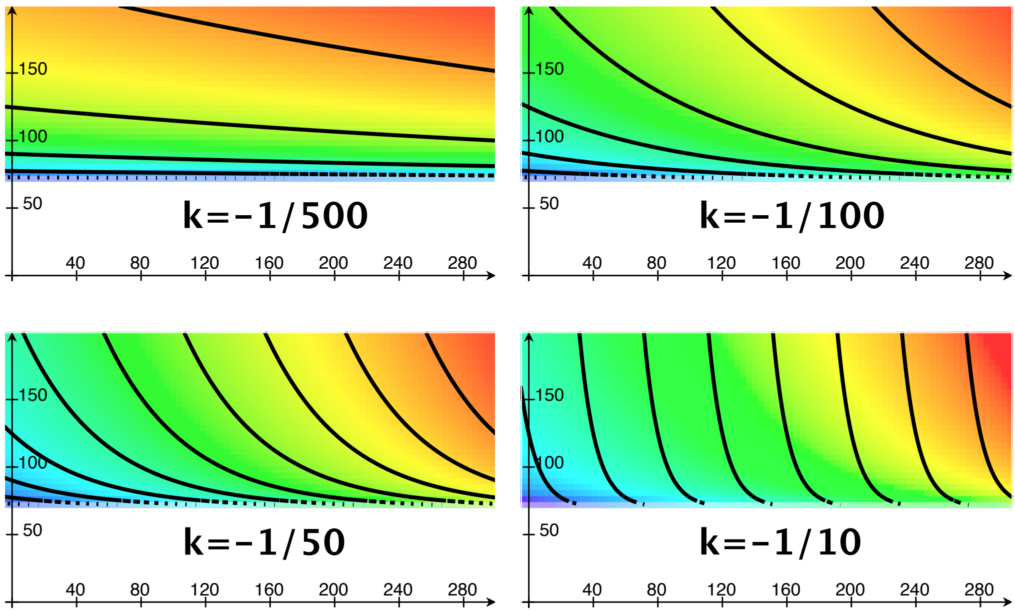
What do we expect from k and A?



IVP: $\frac{dy}{dt} = k(y - 70), \qquad y(0) = 370, \quad y(10) = 340$

General solution: $y = Ae^{kt} + 70$

What do we expect from k and A? $A > 0$, $k < 0$



IVP: $\frac{dy}{dt} = k(y - 70), \qquad y(0) = 370, \quad y(10) = 340$

General solution: $y = Ae^{kt} + 70$

What do we expect from k and A? $A > 0$, $k < 0$, and k small

Step 3: Plug in points and find particular solution

$$370 = y(0) = Ae^0 + 70, \qquad \text{so } \boxed{A = 300}$$

$$340 = y(10) = 300e^{k*10} + 70$$

,

$$\text{so } k = \frac{1}{10}\ln\left(\frac{350 - 70}{300}\right) = \boxed{\ln(.9)/10 \approx -0.0105}.$$

So the particular solution is

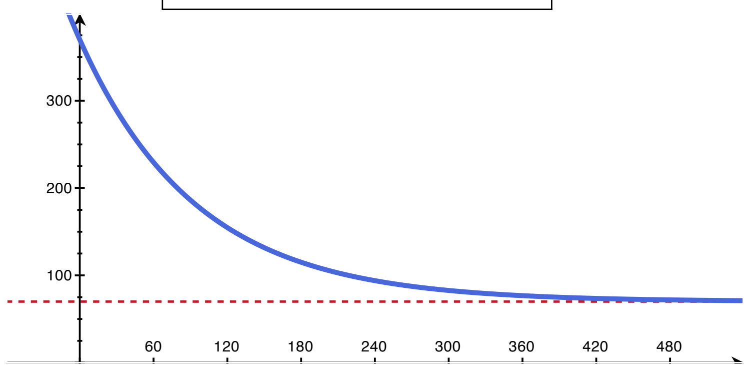
$$\boxed{y = 300e^{t*\ln(.9)/10} + 70}$$

Example 2: Objects heat or cool at a rate proportional to the difference between their temperature and the ambient temperature. Suppose a pie is pulled out of the oven (heated to 370°F), and put into a room that's 70°F. After 10 minutes, the center of the pie is 340°F.

(a) How hot is the pie after 20 minutes?

(b) How long will it take for the center of the pie to cool to 100°F?

Particular solution: $\boxed{y = 300e^{t*\ln(.9)/10} + 70}$



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Particular solution: $\boxed{y = 300e^{t*\ln(.9)/10} + 70}$

Answers:

(a) $y(20) = 300e^{20*\ln(.9)/10} + 70 = \boxed{313}$

(b) $100 = 300e^{t*\ln(.9)/10} + 70$

So $e^{t*\ln(.9)/10} = 30/300 = 1/10$, and so

$$t = \frac{10}{\ln(.9)} \ln(.1) \approx \boxed{218.543}$$

Example 3: The isotope thorium-239 decays at a rate proportional to the amount present, and has a half-life of 24.1 days. How long does 10 grams of thorium-234 take to decay to 1 gram?

"Half-life": The time it takes for an amount of stuff to halve in size.

$$\text{IVP:} \quad \frac{dy}{dt} = ky, \qquad y(0) = 10, \quad y(24.1) = \frac{1}{2}10.$$

Question: What is t when $y(t) = 1$?

To do:
Separate to get general solution;
Plug in points to get specific solution;
Solve $y(t) = 1$ for t

Example 4: The *logistic equation* (another model of population growth)

"A population of rabbits will grow at a rate proportional to the product of the number y of rabbits present and the difference between the population size and the carrying capacity C of their environment."

$$\frac{dy}{dt} = ky(y - C), \qquad y \geq 0 \qquad k < 0$$

What is k? Three cases:

(1) $y < C$: We expect the population to grow, so $\frac{dy}{dt} > 0$. But $y * (y - C) < 0$, so it must be that k is negative!

(2) $y > C$: We expect the population to shrink, so $\frac{dy}{dt} < 0$. But $y * (y - C) > 0$, so it must be that k is still negative!

(3) $y = 0$ or $y = C$: Either way, we expect there to be no growth or decay, so $\frac{dy}{dt} = 0$✓

Solving $\frac{dy}{dt} = ky(y - C)$:

Separate and integrate:

$$\boxed{-\frac{1}{C}\int \frac{1}{y}\, dy + \frac{1}{C}\int \frac{1}{y - C}\, dy = \int \frac{1}{y(y - C)}\, dy = \int k\, dt}$$

Goal: Split $\frac{1}{y(y-C)}$ into the sum of two fractions, each of which we know how to integrate: Find a and b so that

$$\frac{1}{y(y - C)} = \frac{a}{y} + \frac{b}{y - C} = \frac{a(y - C) + by}{y(y - C)} \text{ (common denominator)}$$
$$= \frac{(a + b)y - aC}{y(y - C)}$$

So (compare the coefficients of 1 and y in the numerators)

$$1 = -aC \qquad 0 = a + b \qquad \Longrightarrow a = -\frac{1}{C} \text{ and } b = \frac{1}{C}$$

$$\frac{1}{y(y - C)} = \frac{-\frac{1}{C}}{y} + \frac{\frac{1}{C}}{y - C}$$

Example 4(b):

"Suppose a population of rabbits lives in an environment that will support roughly 100 rabbits at a time. Using the logistic model, how do you expect the rabbit population to grow if you start with 10 rabbits and 6 months later you have 30 rabbits?"'

$$\frac{dy}{dt} = ky(y - 100), \qquad y \geq 0, \quad y(0) = 10, \quad y(0.5) = 30.$$

Step 1: Calculate

$$-\frac{1}{100}\int \frac{1}{y}\ dy + \frac{1}{100}\int \frac{1}{y-100}\ dy = \int k\ dt$$

Step 2: Solve for y.
Hint: simplify using (I) $A \ln|B| = \ln|B^A|$,
(II) $\ln|A| - \ln|B| = \ln|A/B|$, and
(III) $\ln|A| = B$ means $A = \pm e^B$.

Step 3: Plug in $y(0) = 10$ and then $y(0.5) = 30$ to solve for the integration constant and k.

Extra practice: Exponential growth

1. A roast turkey is taken from an oven when its temperature has reached 185° F and is placed on a table in a room where the temperature is 75° F. Assume that it cools at a rate proportional to the difference between its current temperature and the room temperature.

   (a) If the temperature of the turkey is 150° F after half an hour, what is the temperature after 45 min?

   (b) When will the turkey have cooled to 100° F?

2. Radiocarbon dating works on the principle that ^{14}C decays according to radioactive decay with a half life of 5730 years. A parchment fragment was discovered that had about 74% as much ^{14}C as does plant material on earth today. Estimate the age of the parchment.

3. After 3 days a sample of radon-222 decayed to 58% of its original amount.

   (a) What is the half life of radon-222?

   (b) How long would it take the sample to decay to 10% of its original amount?

4. Polonium-210 has a half life of 140 days.

   (a) If a sample has a mass of 200 mg find a formula for the mass that remains after t days.

   (b) Find the mass after 100 days.

   (c) When will the mass be reduced to 10 mg?

   (d) Sketch the graph of the mass function.

5. If the bacteria in a culture increase continuously at a rate proportional to the number present, and the initial number is N_0, find the number at time t.

6. If a radioactive substance disintegrates at a rate proportional to the amount present how much of the substance remains at time t if the initial amount is Q_0?

7. If an object cools at a rate proportional to the the difference between its temperature and the temperature of its surroundings, the initial temperature of the object is T_0 and the temperature of the surroundings are a constant temperature S what is the temperature of the object at time t?

8. Current agricultural experts believe that the worlds farms can feed about 10 billion people. The 1950 world population was 2517 million and the 1992 world population was 5.4 billion. When can we expect to run out of food?

9. Suppose that the GNP in a country is increasing at an annual rate of 4 percent. How many years, at that rate of growth, are required to double the present GNP?

10. What percent of a sample of $^{226}_{88}\text{Ra}$ remains after 100 years? The half life of $^{226}_{88}\text{Ra}$ is 1620 years.

11. A sample contains 4.6 mg of $^{131}_{53}\text{I}$. How many mg will remain after 3.0 days? The half life of $^{131}_{53}\text{I}$ is 8.0 days.

12. The majority of naturally occuring rhenium is $^{187}_{75}\text{Re}$, which is radioactive and has a half life of 7×10^{10} years. In how many years will 5% of the earth's $^{187}_{75}\text{Re}$ decompose?

13. A piece of paper from the Dead Sea scrolls was found to have a $^{14}_{6}\text{C}/^{12}_{6}\text{C}$ ratio 79.5% of that in a plant living today. Estimate the age of the paper, given that the half life of $^{14}_{6}\text{C}$ is 5720 years.

14. The charcoal from ashes found in a cave gave a ^{14}C activity of 8.6 counts per gram per minute. Calculate the age of the charcoal (wood from a growing tree gives a comparable count of 15.3). For ^{14}C the half life is 5720 years.

15. In a certain activity meter, a pure sample of $^{90}_{38}\text{Sr}$ has an activity (rate of decay) of 1000.0 disintegrations per minute. If the activity of this sample after 2.00 years is 953.2 disintegrations per minute, what is the half life of $^{90}_{38}\text{Sr}$.

16. A sample of a wooden artifact from an Egyptian tomb has a $^{14}\text{C}/^{12}\text{C}$ ratio which is 54.2 % of that of freshly cut wood. In approximately what year was the old wood cut? The half life of ^{14}C is 5720 years.