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Exponential Growth and Decay

Warm-up

1. If $(A+B)x - 2A = 3x + 1$ for all x, what are A and B?
   (Hint: if it's true for *all* x, then the coefficients have to match up, i.e. $A + B = 3$ and $-2A = 1$.)
2. Find numbers (maybe not integers) A and B which satisfy
   $$\frac{3x+1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}.$$
   (Hint: Combine the fractions on the right side so that the denominators match on both sides. Then compare the numerators, and see problem 1.)
3. Find numbers (maybe not integers) A and B which satisfy
   $$\frac{1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}.$$
   (Hint: This is like problem 2 again, but when you set up your two equations for A and B, think of 1 like $0 * x + 1$.)
4. Use the last problem to calculate $\int \frac{1}{x^2-4} dx$.

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(Hint: This is like problem 2 again, but when you set up your two equations for A and B, think of 1 like $0 * x + 1$.)

4. Use the last problem to calculate $\int \frac{1}{x^2-4} dx$.

$A + B = 3 \rightarrow B = 3 - A$

$-2A = 1 \rightarrow \boxed{A = -1/2}$; $B = 3 - (-1/2) = \boxed{7/2}$

2. $\frac{A}{x} + \frac{B}{x-2} = \frac{A(x-2) + Bx}{x(x-2)} = \frac{(A+B)x - 2A}{x(x-2)}$

So $\boxed{A = -1/2,\ B = 7/2}$

3. $\frac{A}{(x+2)} + \frac{B}{(x-2)} = \frac{A(x-2) + B(x+2)}{(x+2)(x-2)}$

$= \frac{x(A+B) + 2(B-A)}{(x+2)(x-2)}$

$$\frac{1}{(x+2)(x-2)} = \frac{x(A+B) + 2(B-A)}{(x+2)(x-2)}$$

coef on x: $0 = A + B \longrightarrow B = -A$

coef on 1: $1 = 2(B-A) \longrightarrow 1 = 2(B-(-B))$

$= 4B$

$B = 1/4$

$A = -1/4$

4: $\int \frac{1}{x^2-4}\, dx = \int \frac{1}{(x+2)(x-2)}\, dx$

$= \int -\frac{1}{4}\left(\frac{1}{x+2}\right) + \frac{1}{4}\left(\frac{1}{x-2}\right) dx$

$= -\frac{1}{4}\ln|x+2| + \frac{1}{4}\ln|x-2| + C$

$= \frac{1}{4}\left(\ln|x-2| - \ln|x+2|\right) + C$

$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

Example 1: Suppose a bacteria culture grows at a rate proportional to the number of cells present. If the culture contains 700 cells initially and 900 after 12 hours, how many will be present after 24 hours?

The plan:

1. Put it into math, i.e. Write down an initial value problem.

1'. Look at slope fields to make sure the IVP makes sense.

2. Find the general solution to the IVP.
3. Plug in the points and find the particular solution.
4. Calculate the value of the solution when $t = 24$.

Step 1: Put into math. Initial value problem:

$$\frac{dy}{dt} = ky, \qquad y(0) = 700, \quad y(12) = 900$$

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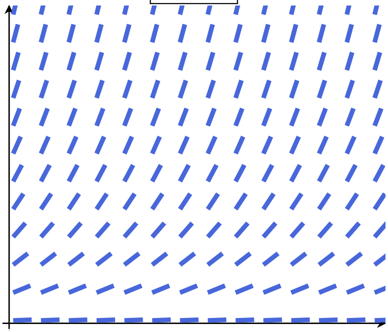
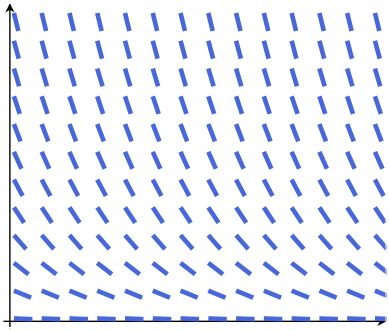
$k > 0$



$k < 0$



$$\boxed{\frac{dy}{dt} = ky, \qquad y(0) = 700, \quad y(12) = 900}$$

Step 2: Find the general solution. To solve: Separate!

$$\int \frac{1}{y} dy = \int k \ dt$$

LHS: $\int \frac{1}{y} \ dy = \ln|y| + c_1$
RHS: $\int k \ dt = kt + c_2$
Putting it together:

$$\ln|y| = kt + C \implies |y| = e^{kt+C} = e^C * e^{kt}$$
$$\implies y = \pm e^C * e^{kt} = Ae^{kt}.$$

General solution: $\boxed{y = Ae^{kt}}$

$$\boxed{\frac{dy}{dt} = ky, \qquad y(0) = 700, \quad y(12) = 900}$$

General solution: $\boxed{y = Ae^{kt}}$

Step 3: Plug in points and find particular solution

$$700 = y(0) = Ae^0 = A, \quad \text{so } y = 700e^{kt}$$

$$900 = 700e^{12k} \implies 12k = \ln(900/700) = ln(9/7)$$

$$\implies k = \frac{1}{12}\ln(9/7) \approx \boxed{0.021}$$

Particular solution: $\boxed{y = 700e^{t * \frac{1}{12}\ln(9/7)}}$

Note: another way to write this is

$$y = 700e^{t * \frac{1}{12}\ln(9/7)} = 700\left(e^{ln(9/7)}\right)^{t/12} = 700\left(\frac{9}{7}\right)^{t/12}$$

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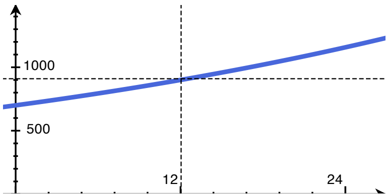
Particular solution: $\boxed{y = 700\left(\frac{9}{7}\right)^{t/12}}$



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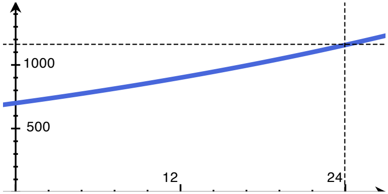


$$y(24) = 700\left(\frac{9}{7}\right)^{24/12} = 700\left(\frac{9}{7}\right)^{2} = 8100/7 \approx 1157.14$$

Example 2: Objects heat or cool at a rate proportional to the difference between their temperature and the ambient temperature. Suppose a pie is pulled out of the oven (heated to 370°F), and put into a room that's 70°F. After 10 minutes, the center of the pie is 340°F.

(a) How hot is the pie after 20 minutes?

(b) How long will it take for the center of the pie to cool to 100°F?

The plan:

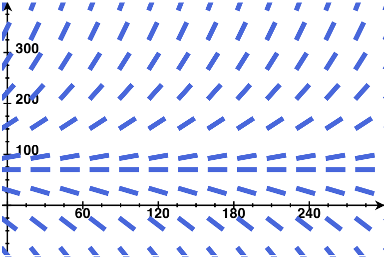
1. Put it into math, i.e. Write down an initial value problem.
2. Find the general solution to the IVP.
3. Plug in the points and find the particular solution.
4. Calculate the value of the solution when $t = 20$.
5. Solve for t when the solution is equal to 100.

Step 1: Put into math. Initial value problem:

$$\frac{dy}{dt} = k(y - 70), \qquad y(0) = 370, \quad y(10) = 340$$

IVP: $\frac{dy}{dt} = k(y - 70), \qquad y(0) = 370, \quad y(10) = 340$

$k > 0$



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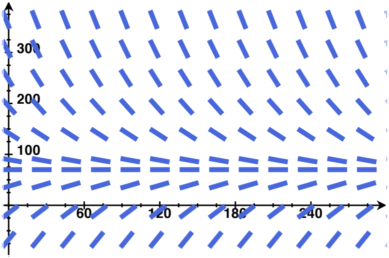


IVP: $\frac{dy}{dt} = k(y - 70), \qquad y(0) = 370, \quad y(10) = 340$

Step 2: Find the general solution. To solve: Separate!

$$\int \frac{1}{y-70} dy = \int k \ dt$$

LHS: $\int \frac{1}{y-70} dy = \ln|y - 70| + c_1$,

RHS: $\int kdt = kt + c_2$

Putting it together: $\ln|y - 70| = kt + c \qquad$ (where $c = c_2 - c_1$).
So

$$y - 70 = \pm e^{kt+c} = \pm e^c * e^{kt} = Ae^{kt} \qquad \text{where } A = \pm e^c,$$

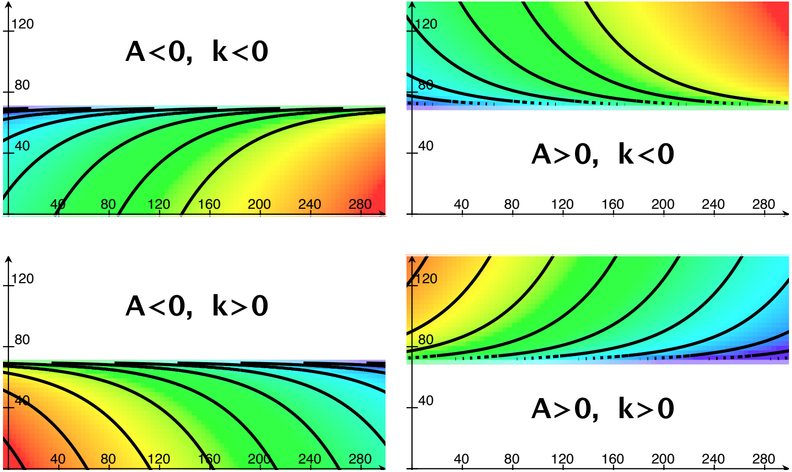
and so

$$\boxed{y = Ae^{kt} + 70}$$

$$\text{IVP: } \frac{dy}{dt} = k(y-70), \qquad y(0)=370, \quad y(10)=340$$

General solution: $y = Ae^{kt} + 70$

What do we expect from k and A?



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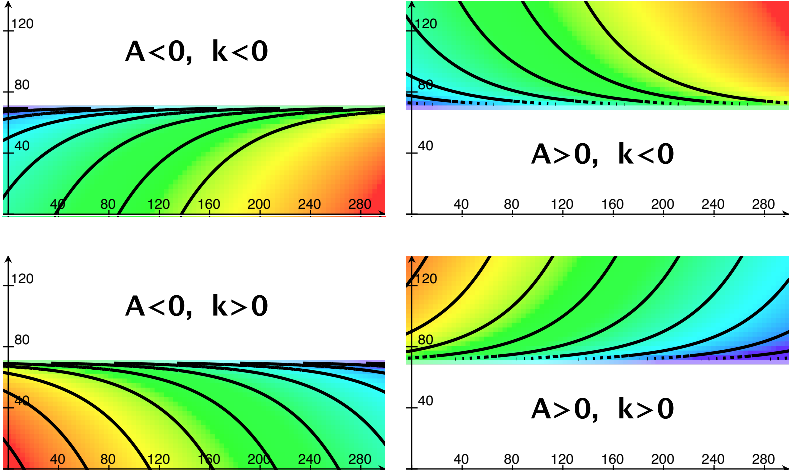
What do we expect from k and A? $A > 0,\ k < 0$



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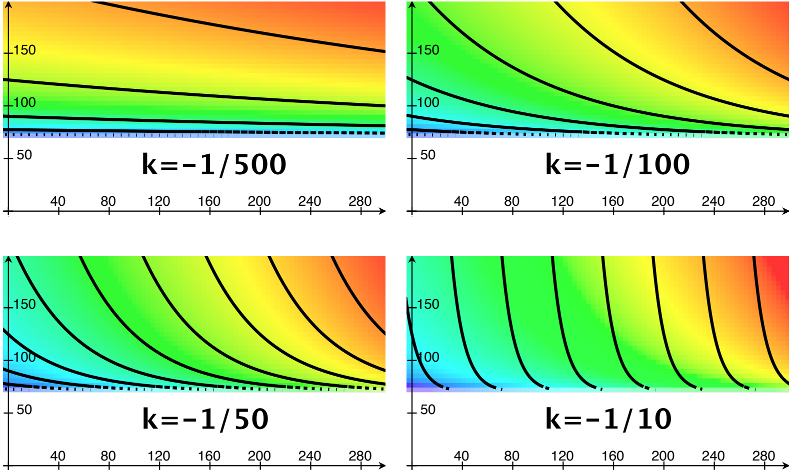

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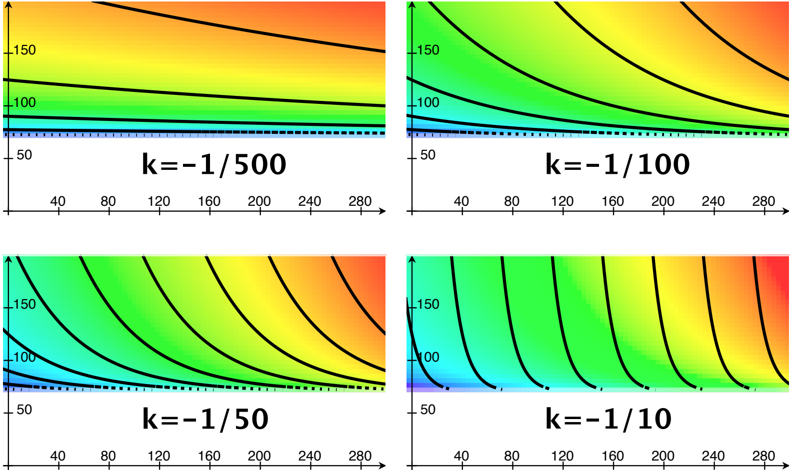

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General solution: $\boxed{y = Ae^{kt} + 70}$

What do we expect from k and A? $A > 0$, $k < 0$, and k small

Step 3: Plug in points and find particular solution

$$370 = y(0) = Ae^0 + 70, \qquad \text{so } \boxed{A = 300}$$

$$340 = y(10) = 300e^{k*10} + 70$$

,

$$\text{so } k = \frac{1}{10}\ln\left(\frac{350-70}{300}\right) = \boxed{\ln(.9)/10 \approx -0.0105}.$$

So the particular solution is

$$\boxed{y = 300e^{t*\ln(.9)/10} + 70}$$

$$= 300\, e^{\ln(9/10)\cdot t/10} + 70$$

$$= \boxed{300\left(9/10\right)^{t/10} + 70}$$

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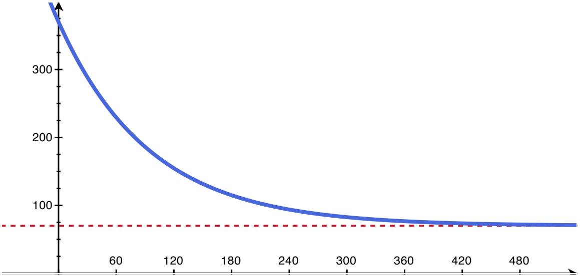

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Answers:

(a) $y(20) = 300e^{20*\ln(.9)/10} + 70 = \boxed{313}$

(b) $100 = 300e^{t*\ln(.9)/10} + 70$

So $e^{t*\ln(.9)/10} = 30/300 = 1/10$, and so

$$t = \frac{10}{\ln(.9)} \ln(.1) \approx \boxed{218.543}$$

Example 3: The isotope thorium-239 decays at a rate proportional to the amount present, and has a half-life of 24.1 days. How long does 10 grams of thorium-234 take to decay to 1 gram?

"Half-life": The time it takes for an amount of stuff to halve in size.

$$\text{IVP:} \quad \frac{dy}{dt} = ky, \qquad y(0) = 10, \quad y(24.1) = \frac{1}{2}10.$$

Question: What is t when $y(t) = 1$?

To do:

Separate to get general solution;
Plug in points to get specific solution;
Solve $y(t) = 1$ for t

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Question: What is t when $y(t) = 1$?

To do:

Separate to get general solution;
Plug in points to get specific solution;
Solve $y(t) = 1$ for t

$y = 0$ or $\int \frac{1}{y}\, dy = \int k\, dt$

$\ln|y| = kt + c$

$\boxed{y = Ae^{kt}}$

$10 = y(0) = A$

$5 = y(24.1) = 10e^{24.1 \cdot k}$

so $k = \frac{1}{24.1} \ln\left(\frac{1}{2}\right)$

$y = 10e^{\ln(1/2) \cdot \frac{1}{24.1} \cdot t} = \boxed{10\left(\frac{1}{2}\right)^{t/24.1}}$

solve $1 = 10 \cdot (1/2)^{t/24.1} \rightarrow \boxed{t = [\log_{1/2}(1/10)] * 24.1}$

Example 4: The *logistic equation* (another model of population growth)

"A population of rabbits will grow at a rate proportional to the product of the number y of rabbits present and the difference between the population size and the carrying capacity C of their environment."

$$\frac{dy}{dt} = ky(y - C), \qquad y \geq 0$$

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$$\frac{dy}{dt} = ky(y - C), \qquad y \geq 0 \qquad k < 0$$

What is k? Three cases:

(1) $y < C$: We expect the population to grow, so $\frac{dy}{dt} > 0$. But $y * (y - C) < 0$, so it must be that k is negative!

(2) $y > C$: We expect the population to shrink, so $\frac{dy}{dt} < 0$. But $y * (y - C) > 0$, so it must be that k is still negative!

(3) $y = 0$ or $y = C$: Either way, we expect there to be no growth or decay, so $\frac{dy}{dt} = 0$✓

Solving $\frac{dy}{dt} = ky(y - C)$:

Separate and integrate:

$$\boxed{-\frac{1}{C}\int \frac{1}{y}\ dy + \frac{1}{C}\int \frac{1}{y-C}\ dy = \int \frac{1}{y(y-C)}\ dy = \int k\ dt}$$

Goal: Split $\frac{1}{y(y-C)}$ into the sum of two fractions, each of which we know how to integrate: Find a and b so that

$$\frac{1}{y(y-C)} = \frac{a}{y} + \frac{b}{y-C} = \frac{a(y-C)+by}{y(y-C)} \quad \text{(common denominator)}$$

$$= \frac{(a+b)y - aC}{y(y-C)}$$

So (compare the coefficients of 1 and y in the numerators)

$$1 = -aC \qquad 0 = a + b \qquad \Longrightarrow \ a = -\frac{1}{C} \text{ and } b = \frac{1}{C}$$

$$\frac{1}{y(y-C)} = \frac{-\frac{1}{C}}{y} + \frac{\frac{1}{C}}{y-C}$$

Example 4(b):

"Suppose a population of rabbits lives in an environment that will support roughly 100 rabbits at a time. Using the logistic model, how do you expect the rabbit population to grow if you start with 10 rabbits and 6 months later you have 30 rabbits?"'

$$\frac{dy}{dt} = ky(y-100), \qquad y \geq 0, \quad y(0) = 10, \quad y(0.5) = 30.$$

Step 1: Calculate

$$-\frac{1}{100}\int \frac{1}{y}\ dy + \frac{1}{100}\int \frac{1}{y-100}\ dy = \int k\ dt$$

Step 2: Solve for y.

Hint: simplify using (I) $A \ln|B| = \ln|B^A|$,

(II) $\ln|A| - \ln|B| = \ln|A/B|$, and

(III) $\ln|A| = B$ means $A = \pm e^B$.

Step 3: Plug in $y(0) = 10$ and then $y(0.5) = 30$ to solve for the integration constant and k.

Example 4(b)

$$-\int \frac{1}{100}\cdot\frac{1}{y}\,dx + \int \frac{1}{100}\,\frac{1}{y-100}\,dy = \int k\,dt$$

$$\frac{1}{100}\left(-\ln|y| + \ln|y-100|\right) = Kt + c$$

$$\frac{1}{100}\ln\left|\frac{y-100}{y}\right| =$$

so $$\frac{y-100}{y} = \pm e^{100(Kt+c)}$$

$$= \pm e^{100c}\cdot e^{100kt} = Ae^{100kt}$$

↳ where $A = \pm e^{100c}$, or 0.

so $y - 100 = yAe^{100kt} \rightarrow y(1 - Ae^{100kt}) = 100$

$$\boxed{y = \frac{100}{1 - Ae^{100kt}}}$$ general soln.

$10 = y(0) = 100/(1-A) \rightarrow 1 - A = 100/10 = 10$

$$\boxed{A = -9}$$

$30 = y(1/2) = 100/1+9e^{100\cdot\frac{1}{2}\cdot k} = 100/1+9e^{50k}$

$1 + 9e^{50k} = 100/30 = 10/3 \qquad e^{50k} = \left(\frac{10}{3}-1\right)/9$

$= 7/27$

$$\boxed{K = \frac{1}{50}\ln(7/27)}$$

$$\boxed{\begin{aligned} y &= 100/(1+9e^{100\cdot\frac{1}{50}\cdot\ln(7/27)\cdot t}) \\ &= 100/(1+9(7/27)^{2t}) \end{aligned}}$$

*notice, since $(7/27)^{2t} \rightarrow 0$ as $t \rightarrow \infty$,

$\lim_{t\to\infty} y(t) = \frac{100}{1+0} = \boxed{100}$ which we expect.