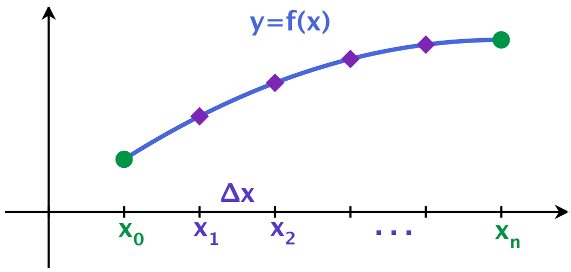
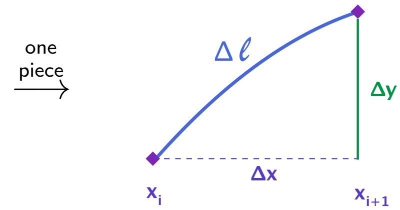
Arc Length

Suppose you want to know what the length of a curve $y = f(x)$ is from the point $(a, f(a))$ to the point $(b, f(b))$:



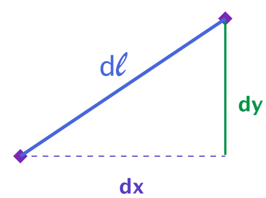


Slice!

Let n go to ∞

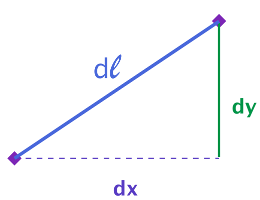
$\downdownarrows$

$$\ell = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta\ell)_i = \int_{x=a}^{x=b} d\ell$$



$$\boxed{d\ell = \sqrt{dx^2 + dy^2}}$$

Manipulating into something we can actually calculate...



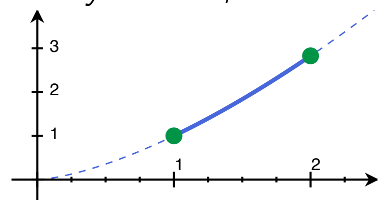
Remember, $y = f(x)$.

$$\begin{aligned} d\ell = \sqrt{dx^2 + dy^2} &= \sqrt{dx^2 + dy^2}\,\frac{dx}{dx} \\ &= \sqrt{\frac{dx^2 + dy^2}{dx^2}}\,dx = \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}}\,dx \\ &= \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= \sqrt{1 + (f'(x))^2}\,dx \end{aligned}$$

So $$\boxed{\ell = \int_{x=a}^{b} \sqrt{1 + (f'(x))^2}\,dx}$$

Example

Find the length of the arc $y = x^{3/2}$, from $x = 1$ to $x = 2$.



$$f(x) = x^{3/2} \quad \Longrightarrow \quad f'(x) = \frac{3}{2}x^{1/2}$$

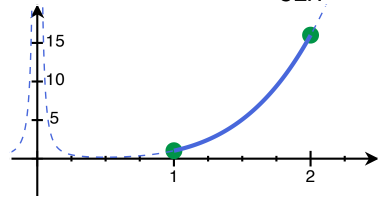
So

$$1 + (f'(x))^2 = 1 + \left(\frac{3}{2}x^{1/2}\right)^2 = 1 + \frac{9}{4}x$$

So

$$\ell = \int_1^2 \sqrt{1 + \frac{9}{4}x}\; dx = \int_1^2 \left(1 + \frac{9}{4}x\right)^{1/2}\; dx$$

Find the length of the curve $y = x^4 + \frac{1}{32x^2}$ from x = 1 to x = 2.



$$f(x) = x^4 + \frac{1}{32}x^{-2} \implies f'(x) = 4x^3 - \frac{1}{16}x^{-3} = \frac{64x^6 - 1}{16x^3}$$

Keeping the algebra tame:
Let $A = (2x)^3 = 8x^3$ and so $A^2 = 64x^6$, and $f'(x) = \frac{A^2-1}{2A}$.
So

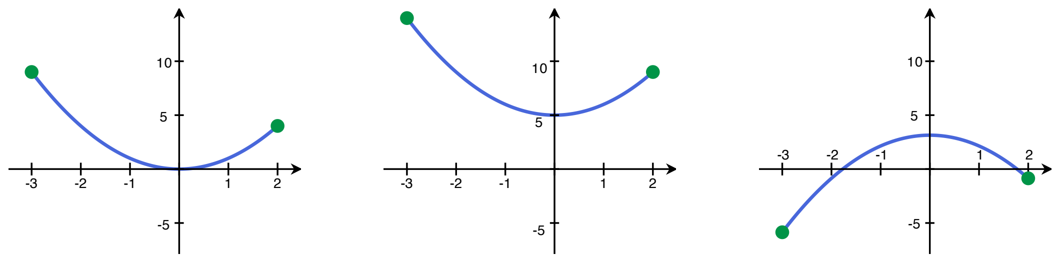
$$1+(f'(x))^2 = 1+\left(\frac{A^2-1}{2A}\right)^2 = 1+\frac{A^4-2A^2+1}{4A^2} = \frac{4A^2+A^4-2A^2+1}{4A^2}$$

$$= \frac{A^4+2A^2+1}{4A^2} = \left(\frac{A^2+1}{2A}\right)^2 = \left(\frac{64x^6+1}{16x^3}\right)^2 = (4x^3 + \tfrac{1}{16}x^{-3})^2$$

Most of the time, the resulting integral is "hard" (not elementary)

Set up (but do not integrate) the integrals which compute the length of the following functions:

1. $f(x) = x^2$ from $x = -3$ to 2
2. $f(x) = x^2 + 5$ from $x = -3$ to 2
3. $f(x) = -x^2 + \pi$ from $x = -3$ to 2
4. $f(x) = \sin(x)$ from $x = 0$ to $\frac{\pi}{2}$
5. $f(x) = e^x$ from $x = 0$ to 1
6. $f(x) = \sqrt{1 - x^2}$ from $x = -1$ to 1



Extra practice: Arclength

1. Use integration to show that the circumference of a circle of radius r is $2\pi r$.

2. Find the length of the curve $y = x^{2/3}$ between $x = -1$ and $x = 8$.

3. Find the length of the curve $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$.

4. Find the length of the curve $y = x^{3/2}$ from $(0, 0)$ to $(4, 8)$.

5. Find the length of the curve $y = (1/3)x^3 + 1/4x$ from $x = 1$ to $x = 3$.

6. Find the length of the curve $y = x^4/4 + 1/8x^2$ from $x = 1$ to $x = 2$.

7. Find the length of the curve $y = (3/5)x^{5/3} - (3/4)x^{1/3}$ from $x = 0$ to $x = 1$.

8. Find the length of the curve $y = (2/3)x^{3/2} - (1/2)x^{1/2}$ from $x = 0$ to $x = 4$.

9. Consider the curve $y = f(x)$, $x \geq 0$, such that $f(0) = a$. Let $s(x)$ denote the arc length along the curve from $(0, a)$ to $(x, f(x))$. Find $f(x)$ if $s(x) = Ax$. What are the permissible values of A?

10. Consider the curve $y = f(x)$, $x \geq 0$, such that $f(0) = a$. Let $s(x)$ denote the arc length along the curve from $(0, a)$ to $(x, f(x))$. Is it possible for $s(x)$ to equal x^n with $n > 1$? Give a reason for your answer.