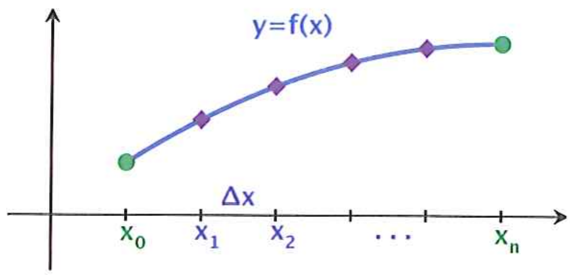
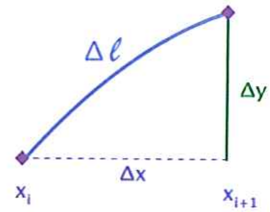
Arc Length

Suppose you want to know what the length of a curve $y = f(x)$ is from the point $(a, f(a))$ to the point $(b, f(b))$:



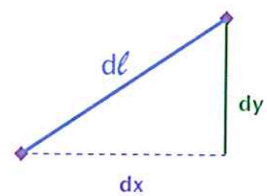
one piece $\longrightarrow$



Slice!

Let n go to ∞

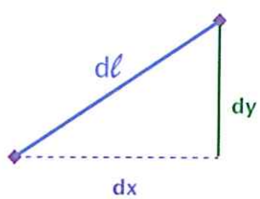
$\downdownarrows$



$$\ell = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta\ell)_i = \int_{x=a}^{x=b} d\ell$$

$$\boxed{d\ell = \sqrt{dx^2 + dy^2}}$$

Manipulating into something we can actually calculate...



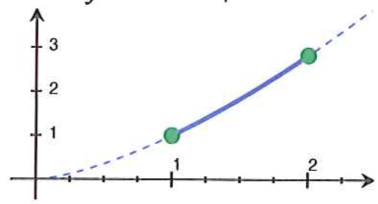
Remember, $y = f(x)$.

$$\begin{aligned} d\ell = \sqrt{dx^2 + dy^2} &= \sqrt{dx^2 + dy^2}\,\frac{dx}{dx} \\ &= \sqrt{\frac{dx^2 + dy^2}{dx^2}}\,dx = \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}}\,dx \\ &= \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= \sqrt{1 + (f'(x))^2}\,dx \end{aligned}$$

So $$\boxed{\ell = \int_{x=a}^{b} \sqrt{1 + (f'(x))^2}\,dx}$$

Example

Find the length of the arc $y = x^{3/2}$, from $x = 1$ to $x = 2$.



$$f(x) = x^{3/2} \quad \Longrightarrow \quad f'(x) = \frac{3}{2}x^{1/2}$$

So

$$1 + (f'(x))^2 = 1 + \left(\frac{3}{2}x^{1/2}\right)^2 = 1 + \frac{9}{4}x$$

So

$$\ell = \int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx = \int_1^2 \left(1 + \frac{9}{4}x\right)^{1/2} dx$$

let $\quad u = 1 + \frac{9}{4}x,$

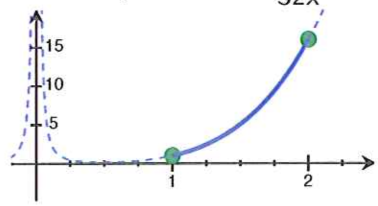
$du = \frac{9}{4}dx \quad , \text{ so } \quad \frac{4}{9}du = dx$

$$\int \left(1+\frac{9}{4}x\right)^{1/2} dx = \frac{4}{9}\int u^{1/2}\,du = \frac{4}{9}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2} + C$$

so $\quad \boxed{\ell = \frac{8}{27}\left(\left(1+\frac{9}{4}\cdot 2\right)^{3/2} - \left(1+\frac{9}{4}\right)^{3/2}\right)}$

Find the length of the curve $y = x^4 + \frac{1}{32x^2}$ from x = 1 to x = 2.



$$f(x) = x^4 + \frac{1}{32}x^{-2} \implies f'(x) = 4x^3 - \frac{1}{16}x^{-3} = \frac{64x^6 - 1}{16x^3}$$

Keeping the algebra tame:
Let $A = (2x)^3 = 8x^3$ and so $A^2 = 64x^6$, and $f'(x) = \frac{A^2-1}{2A}$.
So

$$1+(f'(x))^2 = 1+\left(\frac{A^2-1}{2A}\right)^2 = 1+\frac{A^4-2A^2+1}{4A^2} = \frac{4A^2+A^4-2A^2+1}{4A^2}$$

$$= \frac{A^4+2A^2+1}{4A^2} = \left(\frac{A^2+1}{2A}\right)^2 = \left(\frac{64x^6+1}{16x^3}\right)^2 = (4x^3 + \tfrac{1}{16}x^{-3})^2$$

$$l = \int_1^2 \sqrt{1+(f'(x))^2}\, dx = \int_1^2 4x^3 + \frac{1}{16}x^{-3}\, dx$$

$$= \left[x^4 + \frac{1}{-32}x^{-2}\right]_{x=1}^{2}$$

$$= (2^4 - 1^4) - \frac{1}{32}(2^{-2} - 1^{-2})$$

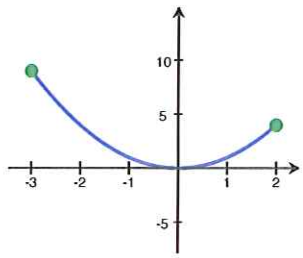
$$= 15 - \frac{1}{32}\left(\frac{1}{4} - 1\right)$$

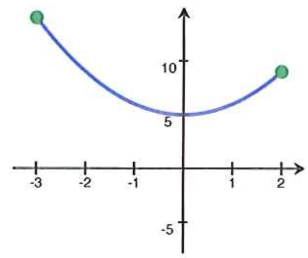
$$= \boxed{15 + \frac{3}{128}}$$

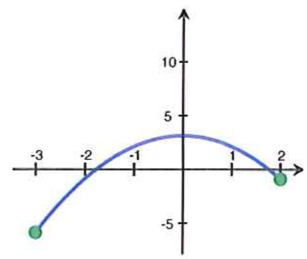
Most of the time, the resulting integral is "hard" (not elementary)

Set up (but do not integrate) the integrals which compute the length of the following functions:

1. $f(x) = x^2$ from $x = -3$ to 2
2. $f(x) = x^2 + 5$ from $x = -3$ to 2
3. $f(x) = -x^2 + \pi$ from $x = -3$ to 2
4. $f(x) = \sin(x)$ from $x = 0$ to $\frac{\pi}{2}$
5. $f(x) = e^x$ from $x = 0$ to 1
6. $f(x) = \sqrt{1 - x^2}$ from $x = -1$ to 1







for 1, 2, and 3, $f' = \pm 2x$, so $\ell = \int_{-3}^{2} \sqrt{1 + 4x^2}\, dx$

4. $\frac{d}{dx} \sin(x) = \cos(x)$ so $\ell = \int_0^{\pi/2} \sqrt{1 + \cos^2(x)}\, dx$

5. $f' = e^x$ so $\ell = \int_0^1 \sqrt{1 + e^{2x}}\, dx$

6. $f' = -2x / 2\sqrt{1-x^2} = -x / \sqrt{1-x^2}$

so $\ell = \int_{-1}^{1} \sqrt{1 + x^2/(1-x^2)}\, dx$

$= \int_{-1}^{1} \sqrt{1/(1-x^2)}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$

$= \arcsin(x)\Big|_{x=-1}^{1} = \boxed{\pi}$