The FUNDAMENTAL Theorem of Calculus
(yay!)

Warm-up

Suppose a particle is traveling at velocity $v(t) = t^2$ from $t = 1$ to $t = 2$. if the particle starts at $y(0) = y_0$,

1. what is the function $y(t)$ which gives the particles position as a function of time (will have a y_0 in it)?
2. how far does the particle travel from $t = 1$ to $t = 2$?

Compare your answer to the upper and lower estimates of the area under the curve $f(x) = x^2$ from $x = 1$ to $x = 2$:

Upper

$$\sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 * \left(\frac{1}{n}\right)$$

Lower

$$\sum_{i=0}^{n-1} \left(1 + \frac{i}{n}\right)^2 * \left(\frac{1}{n}\right)$$

n	Upper	Lower
10	2.485	2.185
100	2.34835	2.31835
1000	2.33483	2.33183

The Fundamental Theorem of Calculus

Theorem (the baby case)

If $F(x)$ is any function satisfying $\frac{d}{dx}F(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Q. What is $\int_1^2 x^2 dx$?

A. $F(x) = \frac{x^3}{3} + C$

So

$$\int_1^2 x^2 dx = F(2) - F(1) = \left(\frac{2^3}{3} + C\right) - \left(\frac{1^3}{3} + C\right)$$

$$= \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3} \approx 2.333}$$

The Fundamental Theorem of Calculus

Theorem (the baby case)

If $F(x)$ is any function satisfying $\frac{d}{dx}F(x) = f(x)$, then

$$\int_a^b f(x)dx = F(x)\Big|_{x=a}^{b} = F(b) - F(a)$$

Q. What is $\int_1^2 x^2 dx$?

A. $F(x) = \frac{x^3}{3}$
So

$$\int_1^2 x^2 dx = \frac{x^3}{3}\Bigg|_{x=1}^{2} = \left(\frac{2^3}{3}\right) - \left(\frac{1^3}{3}\right)$$

$$= \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3} \approx 2.333} \qquad \text{(same answer!)}$$

Examples

Use the fundamental theorem of calculus,

$$\int_a^b f(x)dx = F(b) - F(a)$$

to calculate

1. $\int_2^3 3x\ dx = 3\frac{x^2}{2}\Big|_{x=4}^{6} = 3 * \frac{9}{2} - 3 * \frac{4}{2} = 15/2$
2. $\int_{-1}^1 x^3\ dx = \frac{x^4}{4}\Big|_{x=-1}^{1} = \frac{1^4}{4} - \frac{(-1)^4}{4} = 0$ (odd function!!)
3. $\int_0^\pi \sin(x)\ dx = -\cos(x)\Big|_{x=0}^{\pi} = -\cos(\pi) - (-\cos(0))$
   $= -(-1) - (-1) = 2$
4. $\int_\pi^0 \sin(x)\ dx = -\cos(x)\Big|_{x=\pi}^{0} = -\cos(0) - (-\cos(\pi))$
   $= -(1) - (-(-1)) = -2$

The Fundamental Theorem of Calculus

Theorem (the big case)

If $F(x)$ is any function satisfying $\frac{d}{dt}F(t) = f(t)$, then

$$\int_{a(x)}^{b(x)} f(t)dt = F(t)\Big|_{t=a(x)}^{b(x)} = F(b(x)) - F(a(x))$$

Q. What is $\displaystyle\int_{\sin(x)}^{\ln(x)} t^2 dt$?

A. $F(t) = \dfrac{1}{3}t^3$.

So

$$\int_{\sin(x)}^{\ln(x)} t^2 dt = \frac{1}{3}t^3\Bigg|_{t=\sin(x)}^{\ln(x)} = \left(\frac{1}{3}(\ln(x))^3\right) - \left(\frac{1}{3}(\sin(x))^3\right).$$

Examples

Use the fundamental theorem of calculus,

$$\int_{a(x)}^{b(x)} f(t)dt = F(b(x)) - F(a(x))$$

to calculate

1. $\displaystyle\int_{\sin(x)}^{\cos(x)} 3t\ dt \ = \frac{3}{2}t^2\Big|_{t=\sin(x)}^{\cos(x)} = \frac{3}{2}(\cos(x))^2 - \frac{3}{2}(\sin(x))^2$
2. $\displaystyle\int_{x+1}^{5x^2-3} t^3\ dt \ = \frac{1}{4}t^4\Big|_{t=x+1}^{5x^2-3} = \frac{1}{4}(5x^2-3)^4 - \frac{1}{4}(x+1)^4$
3. $\displaystyle\int_{\arccos(x)}^{0} \sin(t)\ dt \ = -\cos(t)\Big|_{t=\arccos(x)}^{0}$

   $= -\cos(0) - (-\cos(\arccos(x))) = -(1) - (-(x)) = x - 1$

For reference, we calculated $\int_{a(x)}^{b(x)} f(t)\, dt$ where

$$f(t) = t^2 \qquad a(x) = \sin(x) \qquad b(x) = \ln(x).$$

Notice:

$$\frac{d}{dx}\left(\frac{1}{3}(\ln(x))^3 - \frac{1}{3}(\sin(x))^3\right) = \frac{1}{x}(\ln(x))^2 - \cos(x)(\sin(x))^2$$
$$= b'(x)f(b(x)) - a'(x)f(a(x)).$$

In general:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\, dt = b'(x)f(b(x)) - a'(x)f(a(x)).$$

(Don't even have to know $F(t)$!)

Why?

Example: Calculate $\dfrac{d}{dx}\displaystyle\int_{\tan(x)}^{\sin(x)} e^{t^2}\,dt$.

Answer: We can't even calculate $\displaystyle\int e^{t^2}\,dt$!

(There is no elementary function $F(t)$ which satisfies $F'(t) = e^{t^2}$)

But we know $\displaystyle\int e^{t^2}\,dt$ is a function. Call it $F(t)$.

So $\displaystyle\int_{\tan(x)}^{\sin(x)} e^{t^2}\,dt = F(\sin(x)) - F(\tan(x))$.

Therefore
$$\begin{aligned}\frac{d}{dx}\int_{\tan(x)}^{\sin(x)} e^{t^2}\,dt &= \frac{d}{dx}\left(F(\sin(x)) - F(\tan(x))\right)\\ &= \cos(x)F'(\sin(x)) - \sec^2(x)F'(\tan(x))\\ &= \cos(x)f(\sin(x)) - \sec^2(x)f(\tan(x))\\ &= \cos(x)e^{(\sin(x))^2} - \sec^2(x)e^{(\tan(x))^2}\end{aligned}$$

Part 2: Integration by substitution

Warmup

Fill in the blank:

1. Since $\frac{d}{dx}\cos(x^2+1) = \underline{-2x\sin(x^2+1)}$,

$$\text{so } \int \underline{-2x\sin(x^2+1)}\, dx = \cos(x^2+1) + C.$$

2. Since $\frac{d}{dx}\ln|\cos(x)| = \underline{-\frac{\sin(x)}{\cos(x)}}$,

$$\text{so } \int \underline{-\frac{\sin(x)}{\cos(x)}}\, dx = \ln|\cos(x)| + C.$$

(Example: $\frac{d}{dx}x^3 dx = 3x^2$, so $\int 3x^2 dx = x^3 + C$.)

Undoing chain rule

In general:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) * g'(x),$$

so $$\int f'(g(x)) * g'(x) dx = f(g(x)) + C.$$

Example: Calculate the (extremely suggestively written) integral

$$\int \cos(x^3 + 5x - 10) * (3x^2 + 5 * 1 + 0) dx = \sin(x^3 + 5x - 10) + C$$

Check: $\frac{d}{dx} \sin(x^3 + 5x - 10) = \cos(x^3 + 5x - 10) * (3x^2 + 5)$✓

Less obvious chain rules.

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx:

Examples:

$$\int \frac{\cos(x)}{\sin(x)+1}dx = \int \frac{1}{\sin(x)+1} * \cos(x)\ dx$$

$$\int x\sqrt{x^2+1}dx = \int \frac{1}{2}\sqrt{x^2+1} * 2x\ dx$$

$$\int \frac{\cos(\sqrt{x})}{2\sqrt{x}}dx = \int \cos(\sqrt{x}) * \frac{1}{2\sqrt{x}}\ dx$$

Method of Substitution

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

Example: $\displaystyle\int \frac{\cos(x)}{\sin(x)+1}\,dx$

Let $u = g(x)$.	Let $u = \sin(x) + 1$
Calculate du.	$\frac{du}{dx} = \cos(x)$ so $du = \cos(x)dx$
Clear out all of the x's, replacing them with u's.	$\displaystyle\int \frac{1}{u}du$
Calculate the new integral.	$\int \frac{1}{u}du = \ln\|u\| + C$
Substitute back into x's.	$\ln\|u\| + C = \ln\|\sin(x) + 1\| + C$

Check $\frac{d}{dx}\ln|\sin(x)+1| + C = \frac{1}{\sin(x)+1} * \cos(x)$✓

Method of Substitution

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

Example: $\int x\sqrt{x^2+1}\,dx$

Let $u = g(x)$.	Let $u = x^2 + 1$
Calculate $c * du$.	$\frac{du}{dx} = 2x$ so $\frac{1}{2}du = x\ dx$
Clear out all of the x's, replacing them with u's.	$\int \sqrt{u} * \frac{1}{2}du$
Calculate the new integral.	$\frac{1}{2}\int u^{1/2}du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C$
Substitute back into x's.	$= \frac{1}{3}(x^2+1)^{3/2} + C$

Check $\frac{d}{dx}\frac{1}{3}(x^2+1)^{3/2} + C = \frac{1}{3}\frac{3}{2}(x^2+1)^{1/2} * 2x$✓

Give it a try:

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

Example: $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Let $u = g(x)$.	Let $u = \sqrt{x}$
Calculate $c * du$.	$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ so $2du = \frac{1}{\sqrt{x}}\ dx$
Clear out all of the x's, replacing them with u's.	$\int e^u * 2du$
Calculate the new integral.	$2\int e^u du = 2e^u + C$
Substitute back into x's.	$= 2e^{\sqrt{x}} + C$

Check $\frac{d}{dx} 2e^{\sqrt{x}} + C = 2e^{\sqrt{x}} * \frac{1}{2}\frac{1}{\sqrt{x}} \checkmark$