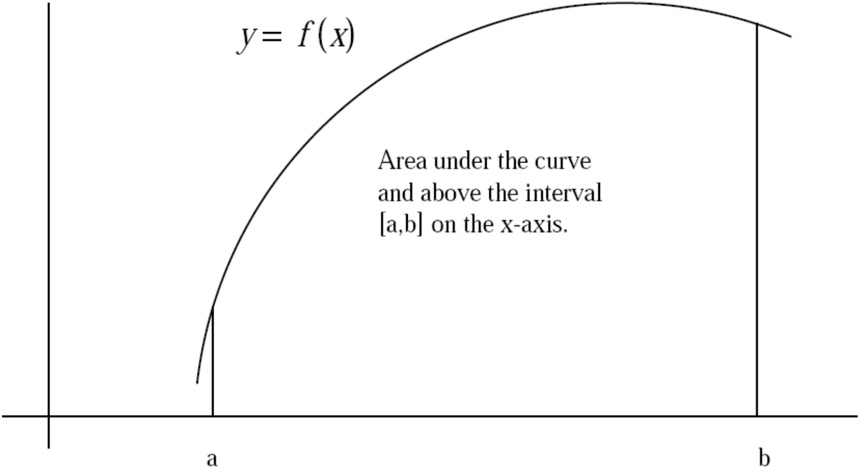
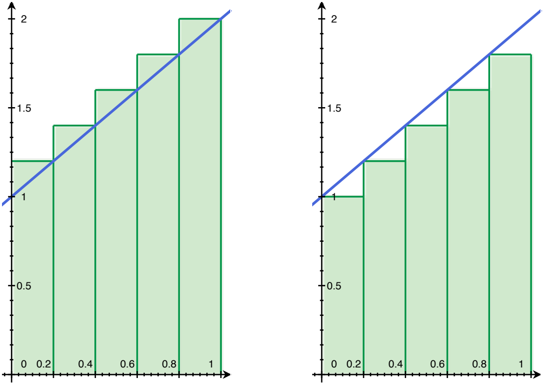
The Definite Integral

The Area Problem



Upper and Lower Sums

Suppose we want to use rectangles to approximate the area under the graph of $y = x + 1$ on the interval $[0, 1]$.



Upper Riemann Sum *Lower Riemann Sum*

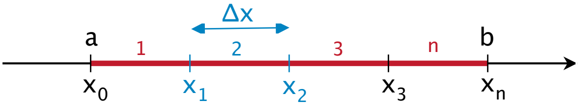
$$31/20 \quad > \quad 1.5 \quad > \quad 29/20$$

As you take more and more smaller and smaller rectangles, if f is nice, both of these will approach the real area.

n	U	L
100	1.505000000	1.495000000
150	1.503333333	1.496666667
200	1.502500000	1.497500000
300	1.501666667	1.498333333
500	1.501000000	1.499000000

In general: finding the Area Under a Curve

Let $y = f(x)$ be given and defined on an interval $[a, b]$.



Break the interval into n equal pieces.

Label the endpoints of those pieces $x_0, x_1, \ldots, x_n$.

Let $\Delta x = x_i - x_{i-1} = \frac{b-a}{n}$ be the width of each interval.

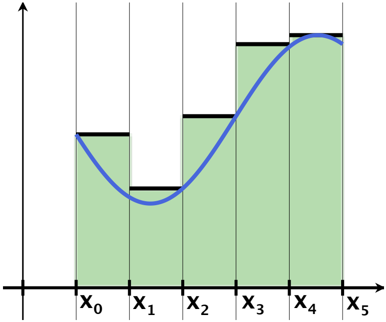
The **Upper Riemann Sum** is: let M_i be the *maximum* value of the function on that i^{th} interval, so

$$U(f, P) = M_1 \Delta x + M_2 \Delta x + \cdots + M_n \Delta x.$$

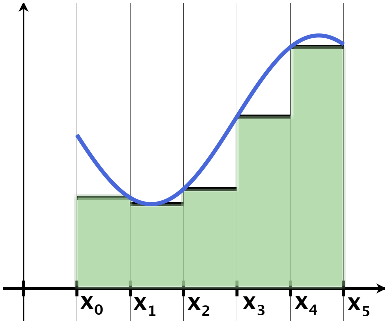
The **Lower Riemann Sum** is: let m_i be the *minimum* value of the function on that i^{th} interval, so

$$L(f, P) = m_1 \Delta x + m_2 \Delta x + \cdots + m_n \Delta x).$$

Take the limit as $n \to \infty$ or $\Delta x \to 0$.



Upper



Lower

Sigma Notation

If m and n are integers with $m \leq n$, and if f is a function defined on the integers from m to n, then the symbol $\sum_{i=m}^{n} f(i)$, called **sigma notation**, is means

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n)$$

Examples:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

$$\sum_{i=1}^{n} \sin(i) = \sin(1) + \sin(2) + \sin(3) + \cdots + \sin(n)$$

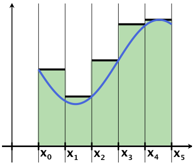
$$\sum_{i=0}^{n-1} x^i = 1 + x + x^2 + x^2 + x^3 + x^4 + \cdots + x^{n-1}$$

The Area Problem Revisited

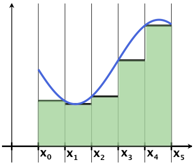
$$\text{Upper Riemann Sum} = \sum_{i=1}^{n} M_i \Delta x$$

$$\text{Lower Riemann Sum} = \sum_{i=1}^{n} m_i \Delta x,$$

where M_i and m_i are, respectively, the maximum and minimum values of f on the ith subinterval $[x_{i-1}, x_i]$, $1 \leq i \leq n$.



Upper

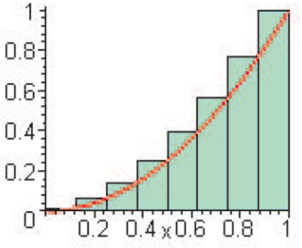


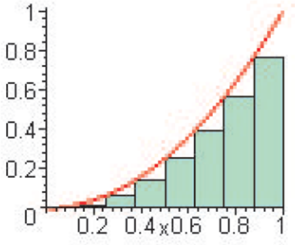
Lower

$\boxed{n = 5}$

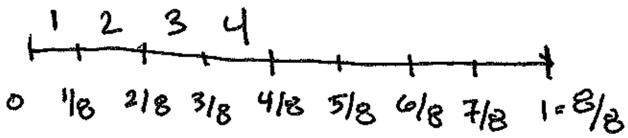
Example

Use an Upper Riemann Sum and a Lower Riemann Sum, first with 8, then with 100 subintervals of equal length to approximate the area under the graph of $y = f(x) = x^2$ on the interval $[0, 1]$.





$n=8$:



mini's happen @ left endpts

max's — " — right "

$$\Delta x = \frac{1-0}{8} = 1/8$$

$$U = \sum_{i=1}^{n} M_i \Delta x \quad : n=8,\ \Delta x = 1/8,\ (\text{left endpt})^2 = M_i$$

$$= \left[\left(\tfrac{1}{8}\right)^2 + (2/8)^2 + (3/8)^2 + (4/8)^2 + (5/8)^2 + (6/8)^2 + (7/8)^2 + (8/8)^2 \right] \cdot \frac{1}{8}$$

↑ $(x_i)^2$ ↑ Δx

b/c $f(x) = x^2$.

$$= \sum_{i=1}^{8} \left(\frac{i}{8}\right)^2 \cdot \frac{1}{8} = \frac{1}{8} \sum_{i=1}^{8} \left(\frac{i}{8}\right)^2$$

$$L = \sum_{i=1}^{8} \left(\frac{i-1}{8}\right)^2 \cdot \frac{1}{8} = \frac{1}{8} \sum_{i=0}^{7} \left(\frac{i}{8}\right)^2$$

$$\left(\frac{0}{8}\right)^2 \cdot \frac{1}{8} + \left(\frac{1}{8}\right)^2 \cdot \frac{1}{8} + \cdots \qquad \frac{1}{8} \cdot \left(\frac{0}{8}\right)^2 + \frac{1}{8}\left(\frac{1}{8}\right)^2 + \cdots$$

$n=100$

$$U = \sum_{i=1}^{100} \left(\frac{i}{100}\right)^2 \frac{1}{100} \qquad L = \sum_{i=1}^{100} \left(\frac{i-1}{100}\right)^2 \cdot \frac{1}{100}.$$

The Definite Integral

We say that f is integrable on $[a, b]$ if there exists a number A such that

$$\text{Lower Riemann Sum} \leq A \leq \text{Upper Riemann Sum}$$

for all n. We write the number as

$$A = \int_a^b f(x)dx$$

and call it the **definite integral** of f over $[a, b]$.

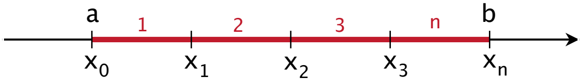
Trickiness: Who wants to find maxima/minima over every interval? Especially as $n \to \infty$? Calculus nightmare!!

More Riemann Sums

Let f be defined on $[a, b]$, and pick a positive integer n.
Let

$$\Delta x = \frac{b-a}{n}$$

Notice:



$$x + 0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad x_3 = a + 3\Delta x, \ldots$$

So let

$$\boxed{x_i = a + i * \Delta x.}$$

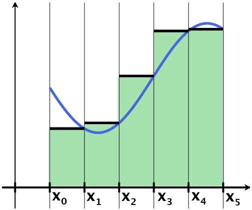
More Riemann Sums

Let f be defined on $[a, b]$, and pick a positive integer n. Let

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i * \Delta x.$$

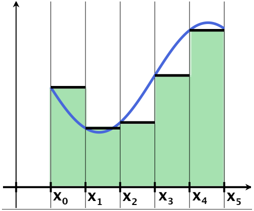
Then the Right Riemann Sum is

$$\sum_{i=1}^{n} f(x_i)\Delta x,$$



and the Left Riemann Sum is

$$\sum_{i=0}^{n-1} f(x_i)\Delta x_i.$$



Integrals made easier

Theorem

If f is "Riemann integrable" on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where c_i is **any** *point in the interval $[x_{i-1}, x_i]$.*

Punchline: We can calculate integrals by just using right or left sums! (instead of upper or lower sums)

Example: Set up left and right limit definitions of $\int_1^4 e^x dx$.
Remember that

> n is the number of pieces we've divided the interval into, and
>
> i indexes the terms in the sum (labels the rectangles).

Each piece:

$$\Delta x = \frac{4-1}{n} = \frac{3}{n} \qquad x_i = 1 + i * \Delta x = 1 + \frac{3i}{n}$$

So, the left Riemann sum is

$$\sum_{i=0}^{n-1} f(x_i)\Delta x = \sum_{i=0}^{n-1} e^{1+\frac{3i}{n}} \left(\tfrac{3}{n}\right) = \frac{3e}{n} \sum_{i=0}^{n-1} \left(e^{3/n}\right)^i$$

and the right Riemann sum is

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} e^{1+\frac{3i}{n}} \left(\tfrac{3}{n}\right) = \frac{3e}{n} \sum_{i=1}^{n} \left(e^{3/n}\right)^i$$

So

$$\int_1^4 e^x dx = \lim_{n\to\infty} \frac{3e}{n} \sum_{i=0}^{n-1} \left(e^{3/n}\right)^i = \lim_{n\to\infty} \frac{3e}{n} \sum_{i=1}^{n} \left(e^{3/n}\right)^i.$$

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So

$$\int_1^4 e^x dx = \lim_{n\to\infty} \frac{3e}{n} \sum_{i=0}^{n-1} \left(e^{3/n}\right)^i = \lim_{n\to\infty} \frac{3e}{n} \sum_{i=1}^{n} \left(e^{3/n}\right)^i.$$

$$e^{1+\frac{3i}{n}} = e^1 \cdot e^{(3/n)\cdot i} = e \cdot \left(e^{3/n}\right)^i$$

$$\sum_{i=1}^{n} e^{1+3i/n}\left(3/n\right) = e\left(e^{3/n}\right)^1 \cdot \frac{3}{n} + e\left(e^{3/n}\right)^2 \cdot \frac{3}{n} + e\left(e^{3/n}\right)^3 \cdot \frac{3}{n} + \cdots$$

$$= e \cdot \frac{3}{n}\left(\left(e^{3/n}\right) + \left(e^{3/n}\right)^2 + \left(e^{3/n}\right)^3 + \cdots \right)$$

$$= e \cdot \frac{3}{n}\left(\sum_{i=1}^{n} \left(e^{3/n}\right)^i \right)$$

On your own:

1. Set up the left limit definition of $\int_{-1}^{5} \sin(x)dx$.

$$\boxed{\lim_{n\to\infty} \sum_{i=0}^{n-1} \sin\left(-1 + \frac{6i}{n}\right)\left(\frac{6}{n}\right)}$$

2. Rewrite the following expressions as $\int_a^b f(x)dx$ by identifying $f(x)$, a, and b. Also, identify if I've used the left or right Riemann sums.

   (a) $\displaystyle\lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\left(6 + \frac{7i}{n}\right)^3 + 2\right)\left(\frac{7}{n}\right).$

   $$\boxed{\text{Left: } \int_6^{13} x^3 + 2\,dx.}$$

   (b) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2 + \frac{i}{n}}{2 - \frac{i}{n}}\left(\frac{1}{n}\right).$

   $$\boxed{\text{Right: } \int_0^{1} \frac{2 + x}{2 - x}\,dx.}$$

On your own:

1. Set up the left limit definition of $\int_{-1}^{5} \sin(x)dx$.

2. Rewrite the following expressions as $\int_a^b f(x)dx$ by identifying $f(x)$, a, and b. Also, identify if I've used the left or right Riemann sums.

   (a) $\lim_{n\to\infty} \sum_{i=0}^{n-1} \left((6+\frac{7i}{n})^3+2\right)\left(\frac{7}{n}\right)$

   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2+\frac{i}{n}}{2-\frac{i}{n}}\left(\frac{1}{n}\right)$

1. $\Delta x = \frac{5-(-1)}{n} = \frac{6}{n}$ $\qquad x_i = a + i\Delta x = -1 + \frac{6i}{n}$

so $\int_{-1}^{5} \sin(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=0}^{n-1} \sin\left(-1+\frac{6i}{n}\right)\cdot\frac{6}{n}.$

↑ LEFT

2. Guess: $\Delta x = \frac{7}{n}$ so $b-a=7 \longrightarrow \underline{b=a+7}$

$x_i = 6+\frac{7i}{n} = 6+i\Delta x \longrightarrow a=6 \longrightarrow b=13.$

$$\boxed{\int_6^{13} x^3+2\,dx}$$

3. Guess: $\Delta x = 1/n \to b-a=1 \longrightarrow b=a+1.$

$x_i = \frac{i}{n}$

$a=0$

$\int_0^1 \frac{2+x}{2-x}dx$

| $x_i = 2+i/n$

| $f(x_i) = \frac{x_i}{4+x_i}$

| $\int_2^3 \frac{x}{4+x}dx$

$2-\frac{i}{n} = 2-\left(2+\frac{i}{n}\right)+2$

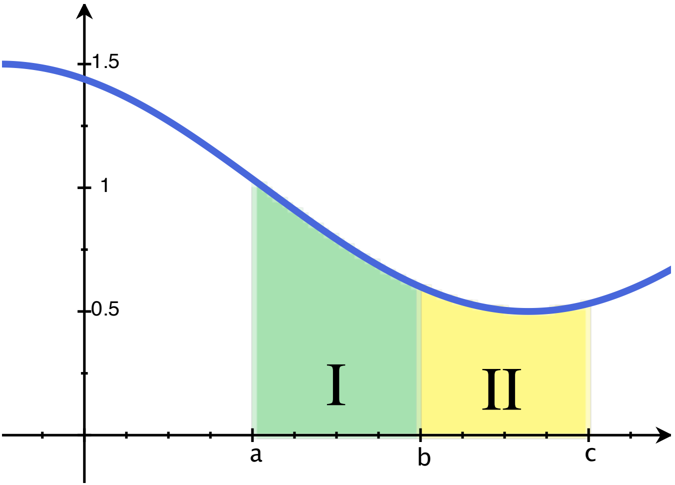
Properties of the Definite Integral

1. $\displaystyle\int_a^a f(x)dx = 0.$

2. If f is integrable and
   (a) $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)dx$ equals the area of the region under the graph of f and above the interval $[a, b]$;
   (b) $f(x) \leq 0$ on $[a, b]$, then $\int_a^b f(x)dx$ equals the **negative** of the area of the region between the interval $[a, b]$ and the graph of f.

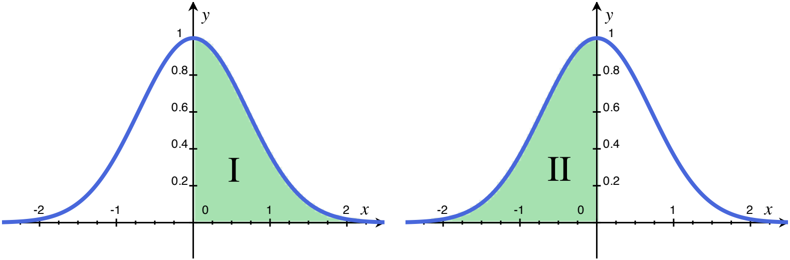
3. $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx.$

4. If $a < b < c$, $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$



5. If f is an **even** function, then

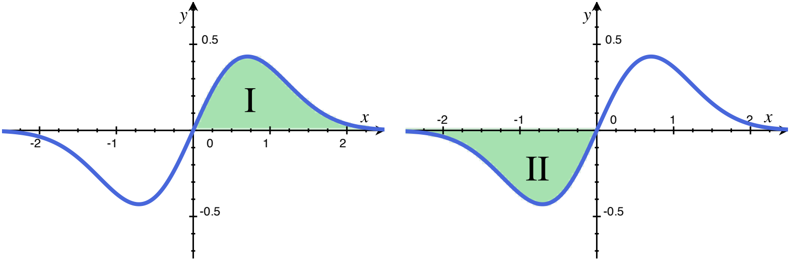
$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$



Area I = Area II

6. If f is an **odd** function, then

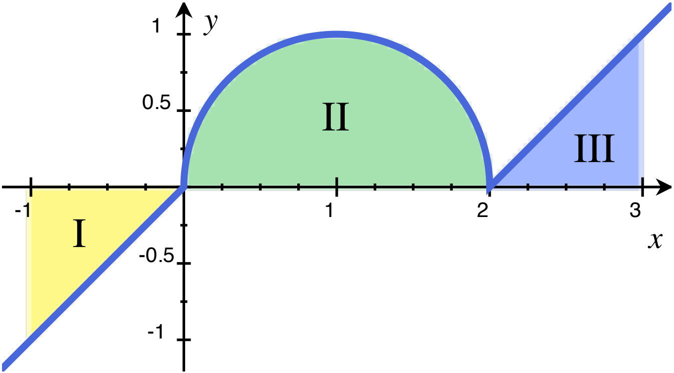
$$\int_{-a}^{a} f(x)dx = 0.$$



Area I = Area II

Example

$$\text{If } f(x) = \begin{cases} x, & x < 0, \\ \sqrt{1-(x-1)^2}, & 0 \geq x \leq 2, \\ x-2, & x \geq 2, \end{cases} \text{ what is } \int_{-1}^{3} f(x)dx?$$



Mean Value Theorem for Definite Integrals

Theorem

Let f be continuous on the interval $[a, b]$. Then there exists c in $[a, b]$ such that

$$\int_a^b f(x)dx = (b-a)f(c).$$

Definition

The *average value* of a continuous function on the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx.$$