Modelling Accumulations

The purpose of calculus is twofold:

1. to find how something is changing, given what it's doing;
2. to find what something is doing, given how it's changing.

We did derivatives

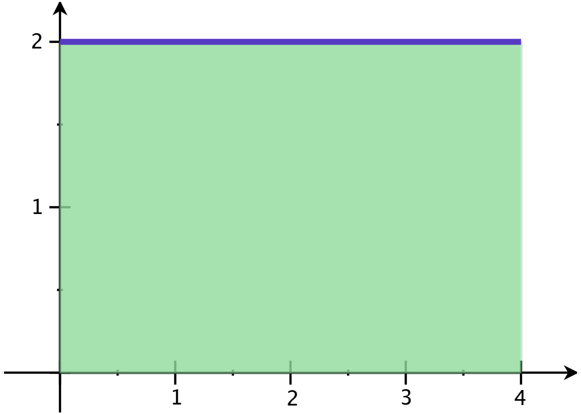
(a) **algebraically** (derivative rules, what is the function?), and
(b) **geometrically** (slopes, increasing/decreasing, what does it look like?)

We did antiderivatives algebraically (what is the function?).
Today: geometric meaning of antiderivatives.

If you travel at 2 mph for 4 hours, how far have you gone?

Answer: 8 miles.

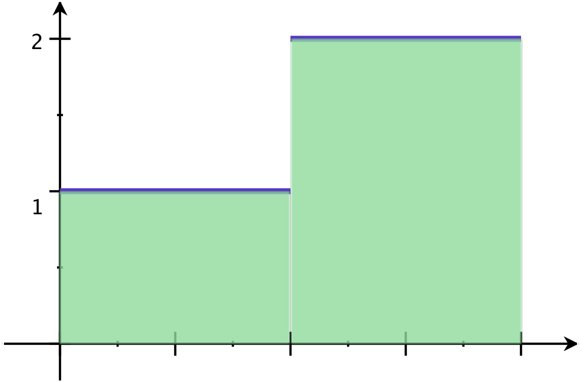
Another way: Area $= \boxed{8}$



(graph of speed, i.e. graph of derivative)

If you travel at 1 mph for 2 hours, and 2 mph for 2 hours, how far have you gone?

Area $= 2+4=\boxed{6}$



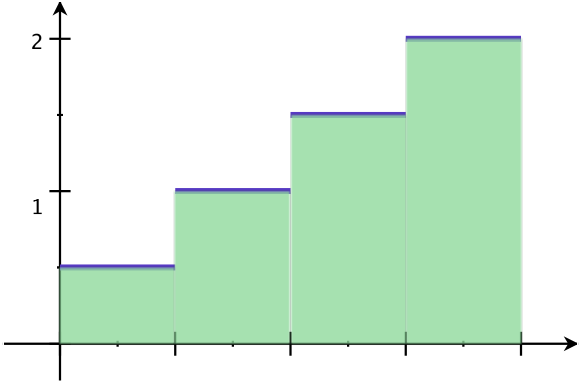
(graph of speed, i.e. graph of derivative)

If you travel at

$$\begin{gathered} .5 \text{ mph for 1 hour,} \\ 1 \text{ mph for 1 hour,} \\ 1.5 \text{ mph for 1 hour,} \\ 2 \text{ mph for 1 hour,} \end{gathered}$$

how far have you gone?

$$\text{Area} = \ .5 + 1 + 1.5 + 2 = \boxed{5}$$



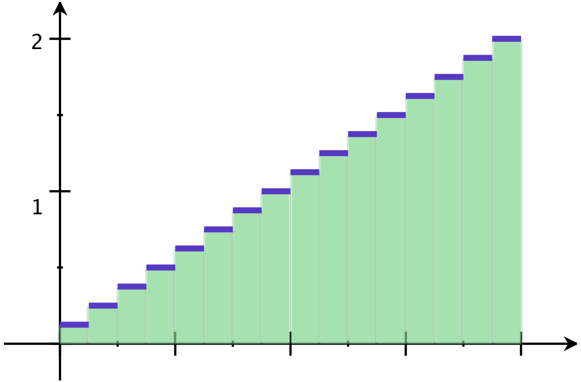
(graph of speed, i.e. graph of derivative)

If you travel at

$$.175 \text{ mph for } 1/4 \text{ hour},$$
$$.25 \text{ mph for } 1/4 \text{ hour},$$
$$\dots$$
$$2 \text{ mph for } 1/4 \text{ hour},$$

how far have you gone?

$$\text{Area} = .175 * .25 + .25 * .25 + \cdots + 2 * .25 = \boxed{4.25}$$



(graph of speed, i.e. graph of derivative)

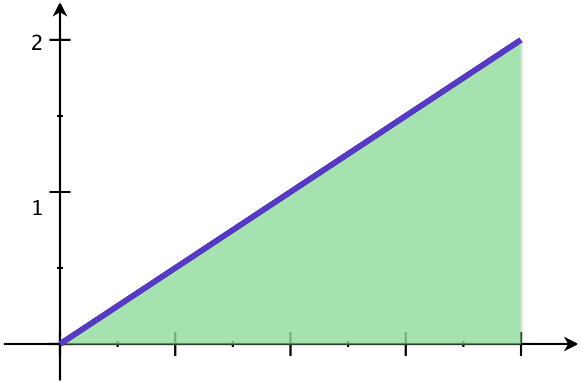
If you travel at $\frac{1}{2}t$ mph for 4 hours, how far have you gone?

Check our answer using antiderivatives from last time:

$$\text{position} = s(t) = \int \frac{1}{2}t \; dt = \frac{1}{4}t^2 + C$$

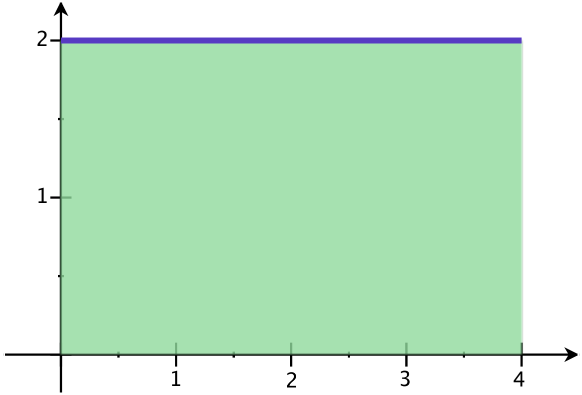
So distance $= s(4) - s(0) = \frac{1}{4} * 16 + C - (\frac{1}{4} * 0 + C) = 4$ ✓

Area = $\boxed{4}$ (it's a triangle)



(graph of speed, i.e. graph of derivative)

Choose another sequence of speeds:



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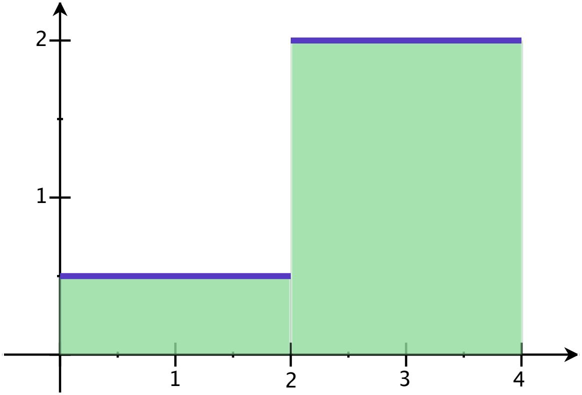

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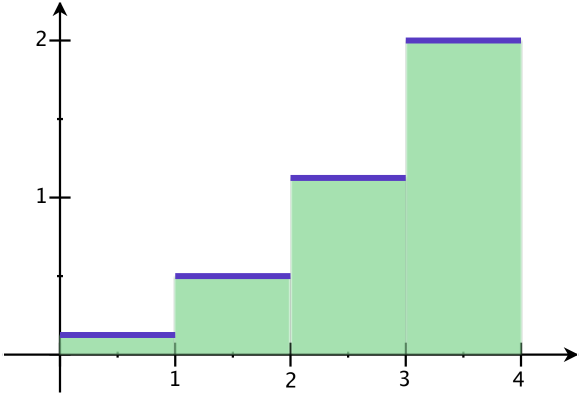

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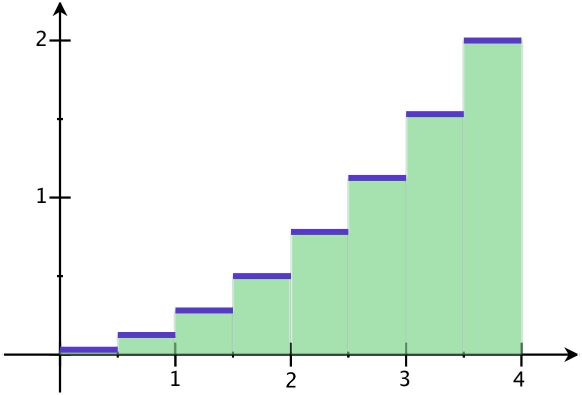

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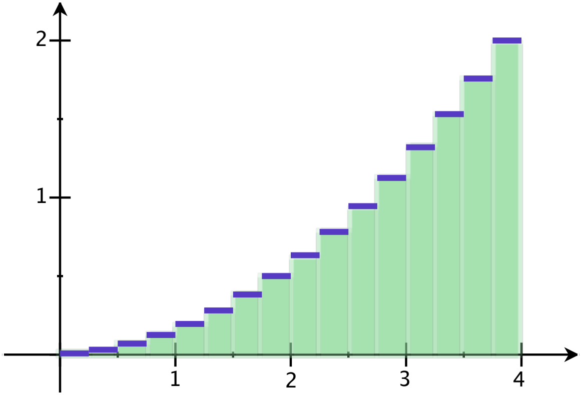

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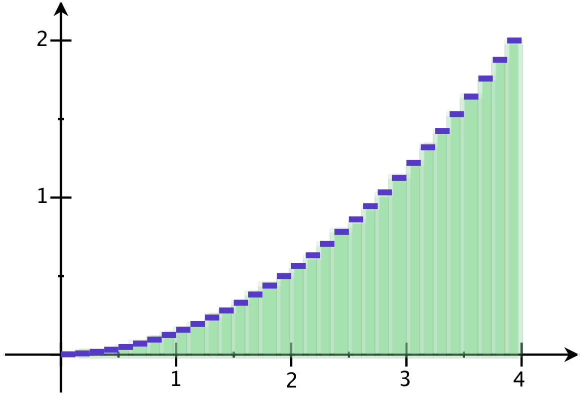
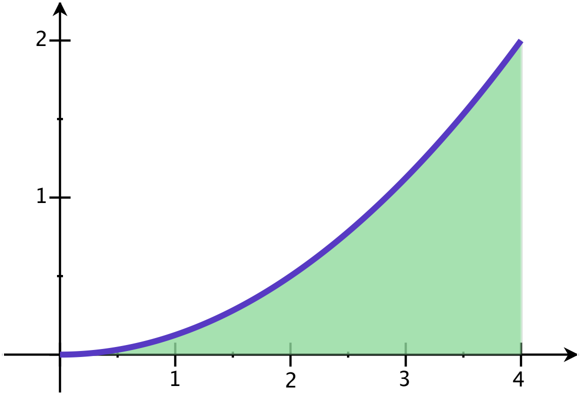
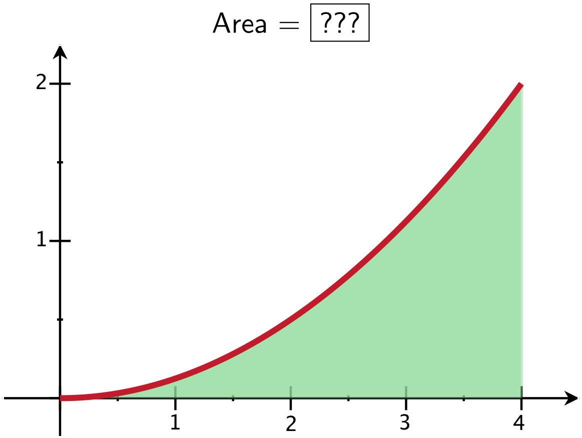


Choose another sequence of speeds:

$y = \frac{1}{8}x^2$, Area = $\boxed{???}$



Estimate the area under the curve $y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:



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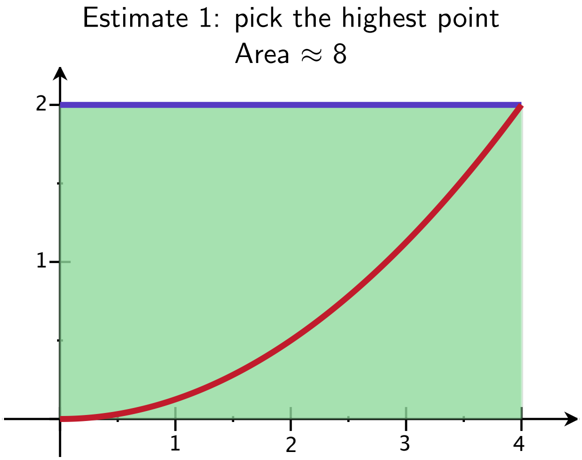
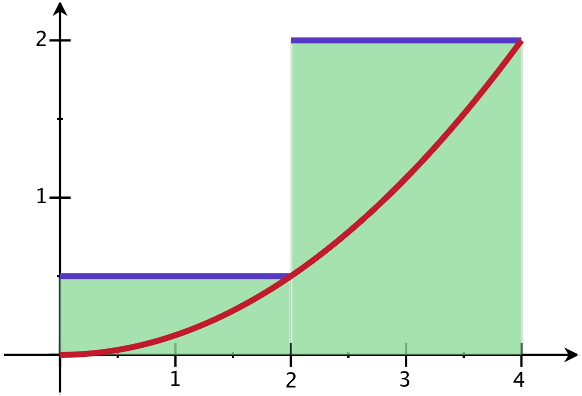


Estimate the area under the curve $y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

Estimate 2: pick two points

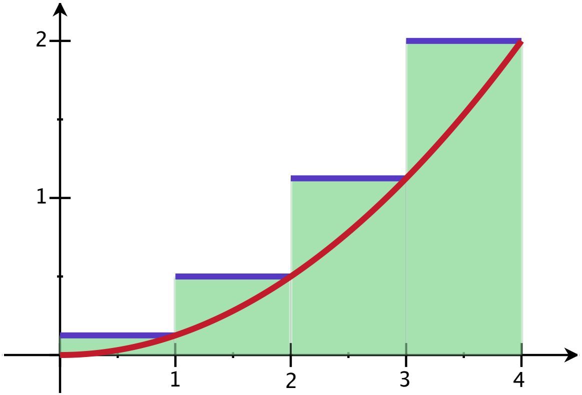
Area $\approx 1+4 = \boxed{5}$



Estimate the area under the curve $y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

Estimate 3: pick four points

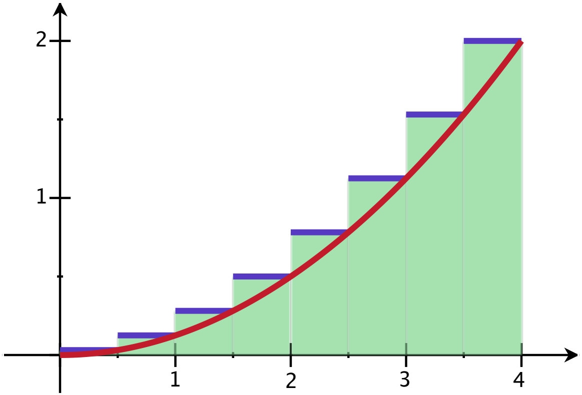
Area $\approx \frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 2 = \boxed{3.75}$



Estimate the area under the curve $y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

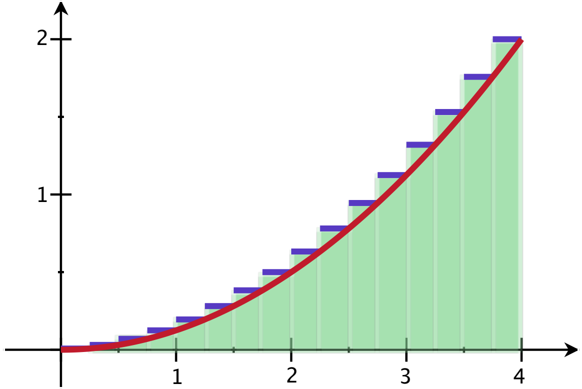
Estimate 4: pick eight points

$$\text{Area} \approx \frac{1}{32} * \frac{1}{2} + \frac{1}{8} * \frac{1}{2} + \cdots + 2 * \frac{1}{2} = \boxed{3.1875}$$

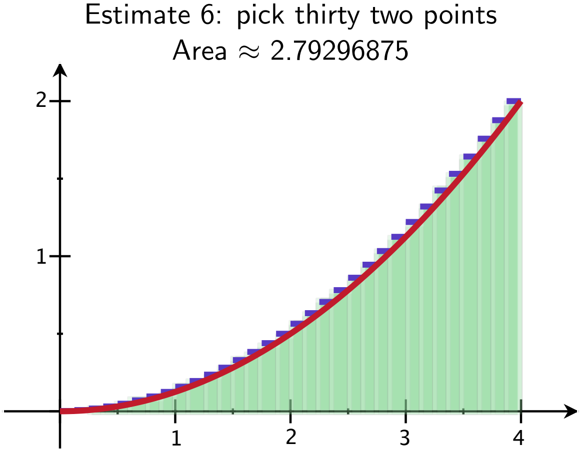


Estimate the area under the curve
$y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

Estimate 5: pick sixteen points
Area ≈ 2.921875



Estimate the area under the curve
$y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:



Compare to

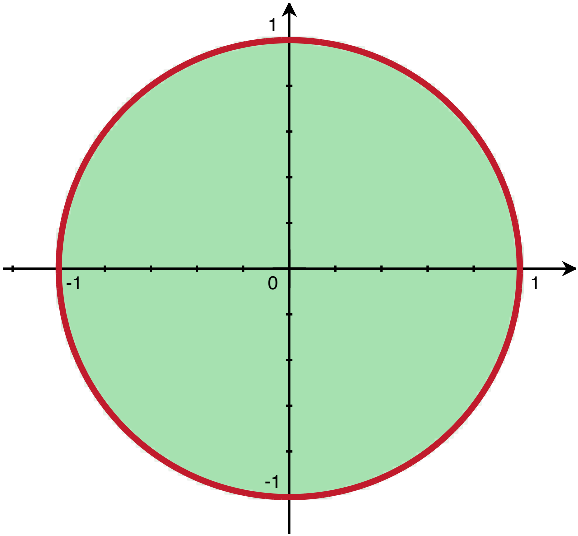
$$s(t) = \int \frac{1}{8} t^2 \, dt = \frac{1}{8} \cdot \frac{1}{3} t^3 + C$$

$$s(4) - s(0) = \frac{1}{8 \cdot 3} \left(4^3 - 0^3 \right) + C - C$$

$$= \frac{4^3}{8 \cdot 3} = \frac{4 \cdot 2 \cdot 2 \cdot 4}{4 \cdot 2 \cdot 3} = \boxed{\frac{8}{3}}$$

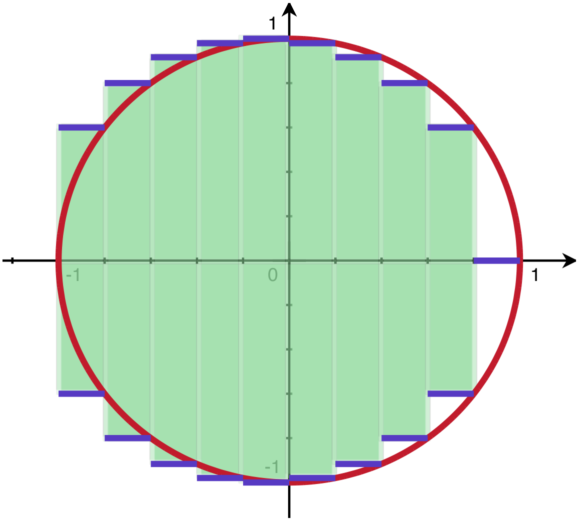
$$= 2.\overline{6}$$

Estimating the Area of a Circle with $r = 1$



Estimating the Area of a Circle with $r = 1$

Divide it up into rectangles:



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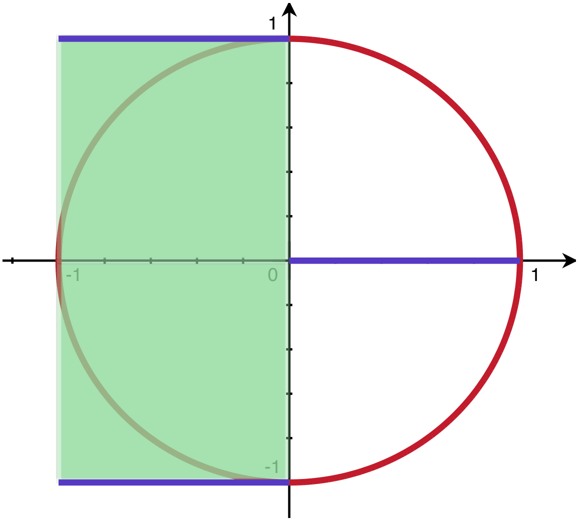
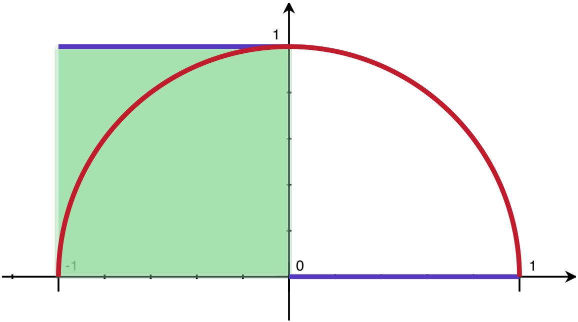


Estimating the Area of a Circle with $r = 1$

Divide it up into rectangles:

Estimate area of the half circle ($f(x) = \sqrt{1 - x^2}$) and mult. by 2.



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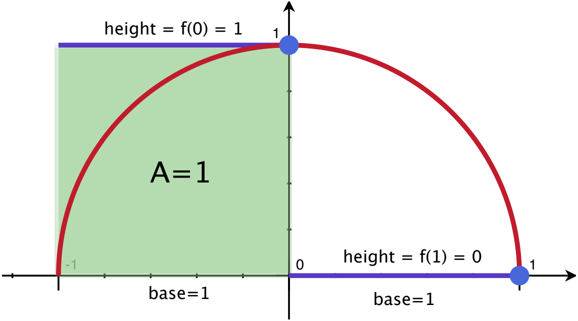

# rect.	Area
4	2*1 = 2
4*2	
4*3	
4*4	
4*5	

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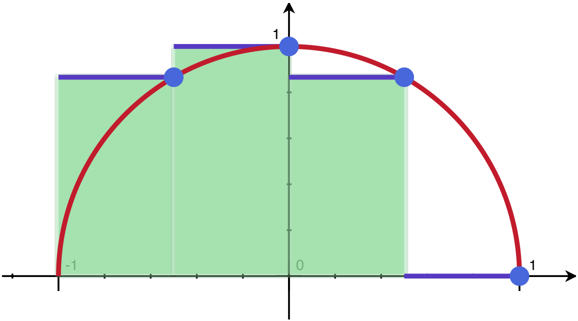

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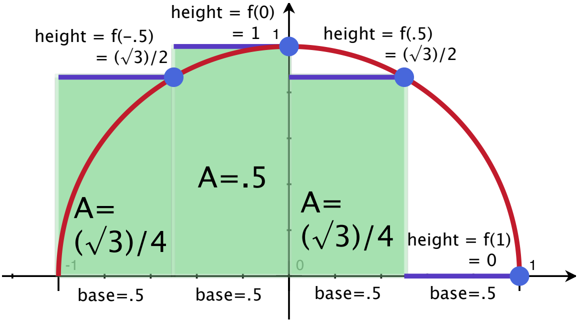

# rect.	Area
4	$2*1 = 2$
4*2	$\sqrt{3} + 1 \approx 2.732$
4*3	
4*4	
4*5	

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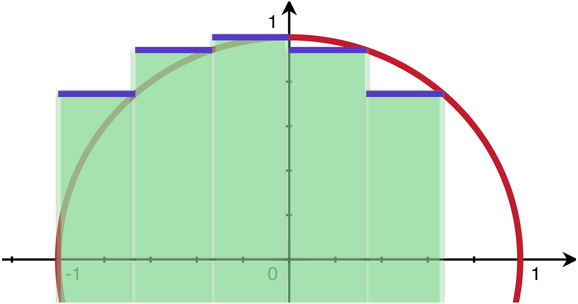

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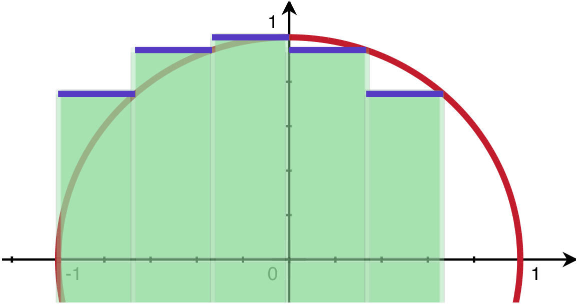

# rect.	Area
4	$2*1 = 2$
4*2	$\sqrt{3} + 1 \approx 2.732$
4*3	$\frac{2}{3} + \frac{8\sqrt{2}+4\sqrt{5}}{9} \approx 2.918$
4*4	
4*5	

Estimating the Area of a Circle with $r = 1$

Divide it up into rectangles:

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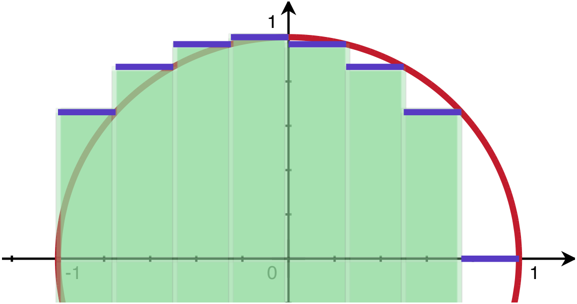

# rect.	Area
4	$2*1 = 2$
4*2	$\sqrt{3} + 1 \approx 2.732$
4*3	$\frac{2}{3} + \frac{8\sqrt{2}+4\sqrt{5}}{9} \approx 2.918$
4*4	2.996
4*5	

Estimating the Area of a Circle with $r = 1$

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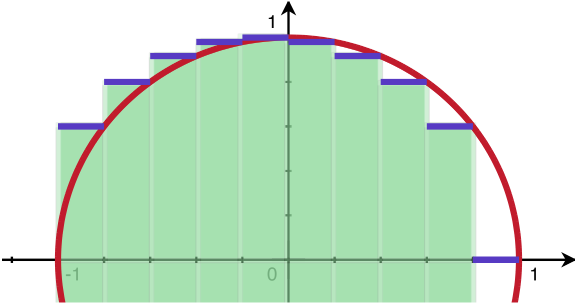

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4	$2*1 = 2$
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4*3	$\frac{2}{3} + \frac{8\sqrt{2}+4\sqrt{5}}{9} \approx 2.918$
4*4	2.996
4*5	3.037

Estimating the Area of a Circle with $r = 1$

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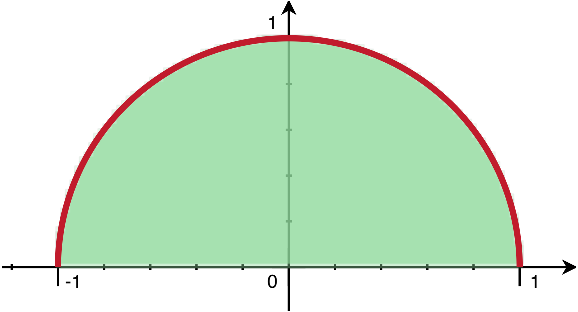

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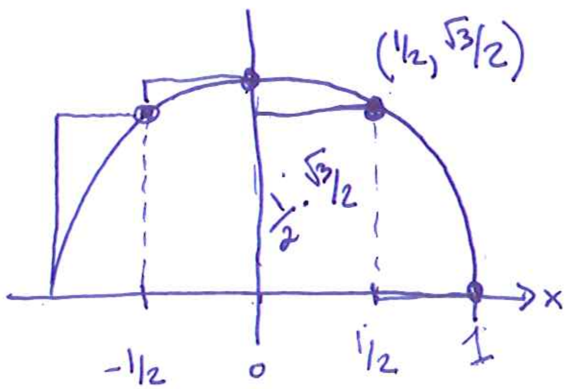
4*3	$\frac{2}{3} + \frac{8\sqrt{2}+4\sqrt{5}}{9} \approx 2.918$
4*4	2.996
4*5	3.037
4*100	3.140

Draw pictures and estimate area for 4*2 and 4*3 rectangles:

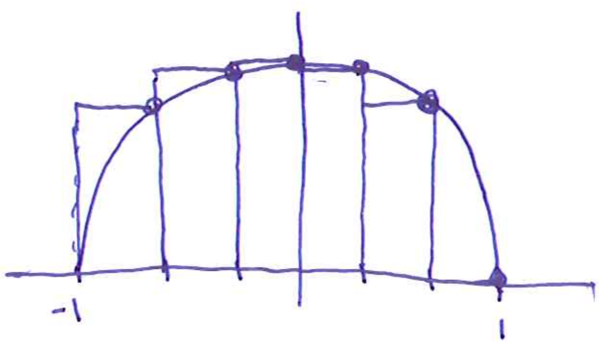


x	$f(x)$	$a(\text{rect})$
1	0	0
$1/2$	$\sqrt{1-1/4} = \sqrt{3}/2$	$\sqrt{3}/4$
0	1	$1/2$
$-1/2$	$\sqrt{3}/2$	$\sqrt{3}/4$

$f(x) = \sqrt{1-x^2}$

solve $x^2+y^2=1$ for y

$$\text{Area} = 2 * \left(0 + \frac{\sqrt{3}}{4} + \frac{1}{2} + \frac{\sqrt{3}}{4}\right) = \boxed{\sqrt{3}+1}$$



x	$f(x)$	$a(\text{rect})$
1	0	0
$2/3$	$\sqrt{1-4/9} = \sqrt{5}/3$	$\sqrt{5}/9$
$1/3$	$\sqrt{1-1/9} = \sqrt{8}/3$	$2\sqrt{2}/9$
0	1	$1/3$
$-1/3$	$\sqrt{8}/3 = \frac{2\sqrt{2}}{3}$	$2\sqrt{2}/9$
$-2/3$	$\sqrt{5}/3$	$\sqrt{5}/9$

$$\text{Area} = 2 * \left(0 + 2 * \sqrt{5}/9 + 2 * 2\sqrt{2}/9 + 1/3\right)$$

Numerical Integration

Big idea: Estimating, and then taking a limit.

Let the number of pieces go to ∞
i.e. let the base of the rectangle for to 0.

Good for:

1. Approximating accumulated change when the antiderivative is unavailable.
2. Making precise the notion of 'area' (we'll also to lengths and volumes)

Example: estimating volume using data

A small dam breaks on a river. The average flow out of the stream is given by the following:

hours	m^3/s	hours	m^3/s	hours	m^3/s
0	150	4.25	1460	8.25	423
0.25	230	4.5	1350	8.5	390
0.5	310	4.75	1270	8.75	365
0.75	430	5	1150	9	325
1	550	5.25	1030	9.25	300
1.25	750	5.5	950	9.5	280
1.5	950	5.75	892	9.75	260
1.75	1150	6	837	10	233
2	1350	6.25	770	10.25	220
2.25	1550	6.5	725	10.5	199
2.5	1700	6.75	658	10.75	188
2.75	1745	7	610	11	180
3	1750	7.25	579	11.25	175
3.25	1740	7.5	535	11.5	168
3.5	1700	7.75	500	11.75	155
3.75	1630	8	460	12	150
4	1550				

Example: estimating volume using data

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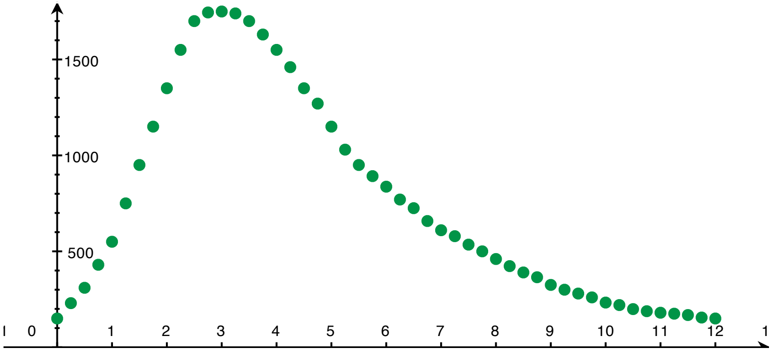
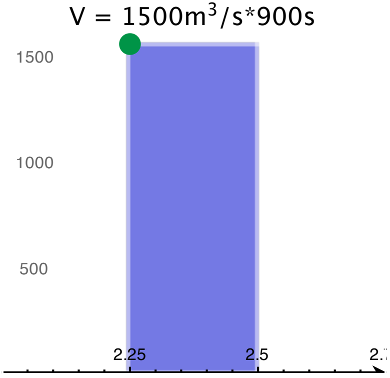


Over each time interval, we estimate the volume of water by

Average rate $\times$ 900 s



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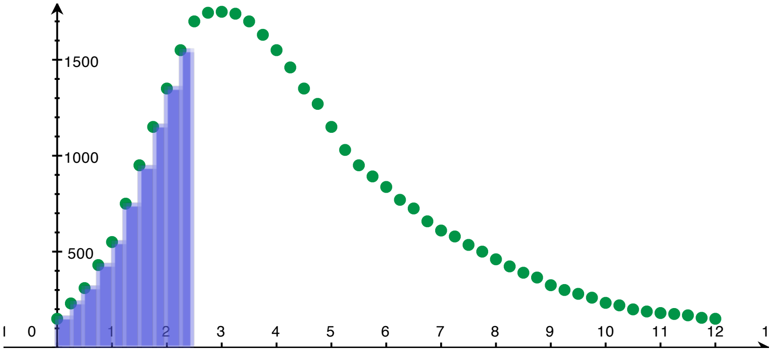


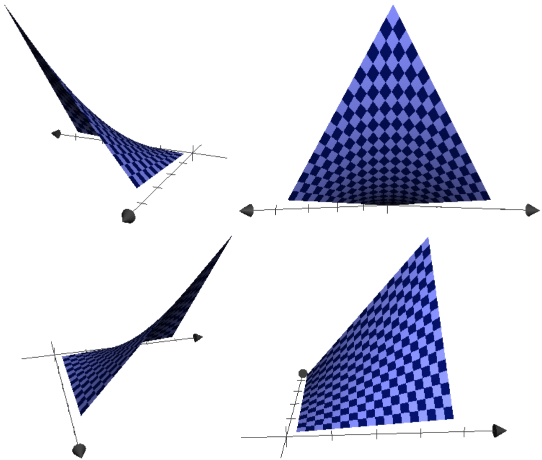
Over each time interval, we estimate the volume of water by

$$\text{Average rate} \times 900 \text{ s}$$

hours	m^3	hours	m^3	hours	m^3
0	135000	4.25	1314000	8.25	380700
0.25	207000	4.5	1215000	8.5	351000
0.5	279000	4.75	1143000	8.75	328500
0.75	387000	5	1035000	9	292500
1	495000	5.25	927000	9.25	270000
1.25	675000	5.5	855000	9.5	252000
1.5	855000	5.75	802800	9.75	234000
1.75	1035000	6	753300	10	209700
2	1215000	6.25	693000	10.25	198000
2.25	1395000	6.5	652500	10.5	179100
2.5	1530000	6.75	592200	10.75	169200
2.75	1570500	7	549000	11	162000
3	1575000	7.25	521100	11.25	157500
3.25	1566000	7.5	481500	11.5	151200
3.5	1530000	7.75	450000	11.75	139500
3.75	1467000	8	414000	12	135000
4	1395000			total=33,319,800	

Example: estimating volume under a function of 2 variables

A tent is raised and has height given by xy over the 2×2 grid where $0 < x < 2$ and $0 < y < 2$. What is the volume of the tent?



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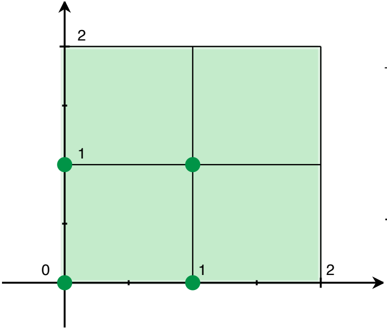
Estimate via boxes!
Volume $=$ base *height.



x	y	height $= xy$	volume
0	0	0	0 * 1
0	1	0	0 * 1
1	0	0	0 * 1
1	1	1	1 * 1

total volume ≈ 1

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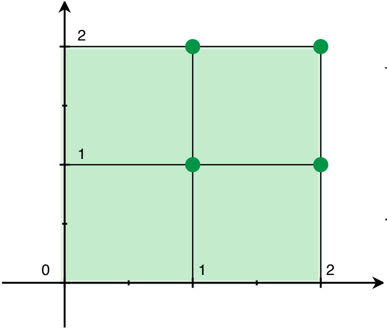


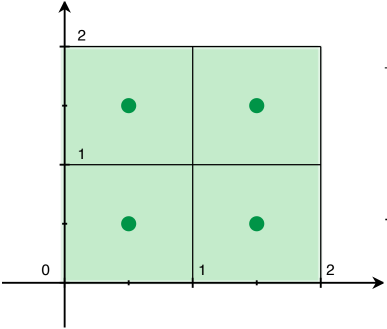
x	y	height $= xy$	volume
1	1	1	1 * 1
1	2	2	2 * 1
2	1	2	2 * 1
2	2	4	4 * 1

total volume ≈ 9

Example: estimating volume under a function of 2 variables

A tent is raised and has height given by xy over the 2×2 grid where $0 < x < 2$ and $0 < y < 2$. What is the volume of the tent?

Estimate via boxes!
Volume = base *height.

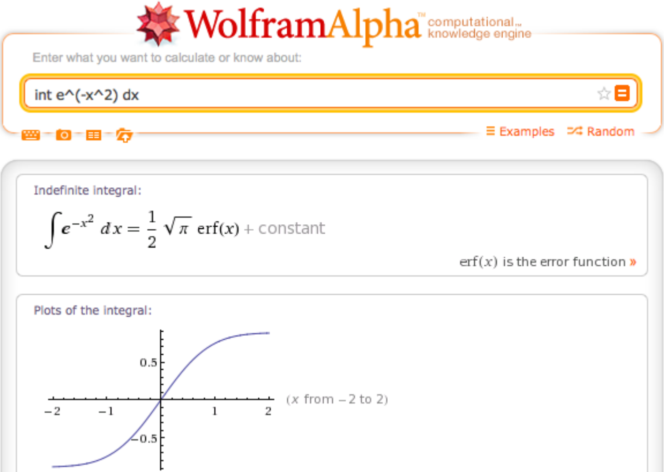


x	y	height $= xy$	volume
.5	.5	.25	.5 * 1
.5	1.5	.75	.75 * 1
1.5	.5	.75	.75 * 1
1.5	1.5	2.25	2.25 * 1

total volume ≈ 4.25

Example: functions without nice antiderivatives

What is $\int e^{-x^2} dx$?

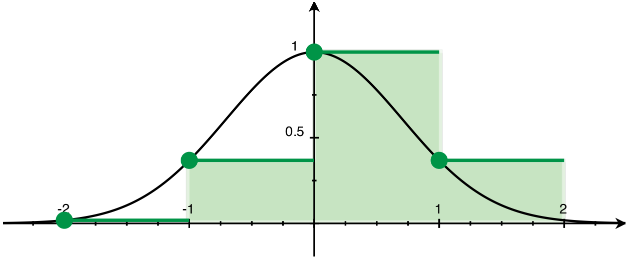


From Wikipedia: "In mathematics, the error function (also called the Gauss error function) is a special function (non-elementary) of sigmoid shape which occurs in probability, statistics and partial differential equations. "

Other methods of numerical integration

We did rectangles, but we could use other shapes (that we know how to integrate under) to better represent the shape of the function.

$f(x) = e^{-x^2}$ between $x = -2$ and $x = 2$: $A = 1.76416\ldots$



For n pieces,

for **rectangles** pick n points to draw n rectangles; $A \approx 1.75407\ldots$

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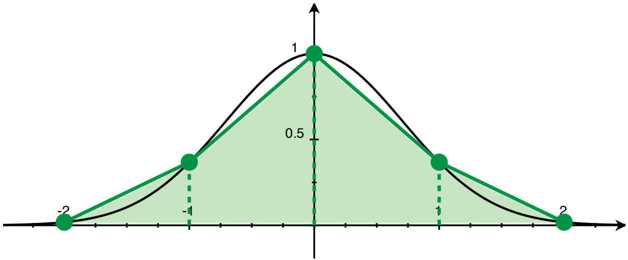

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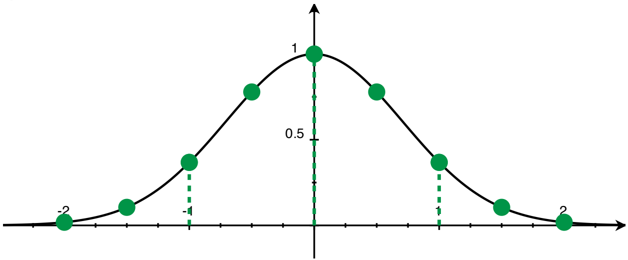

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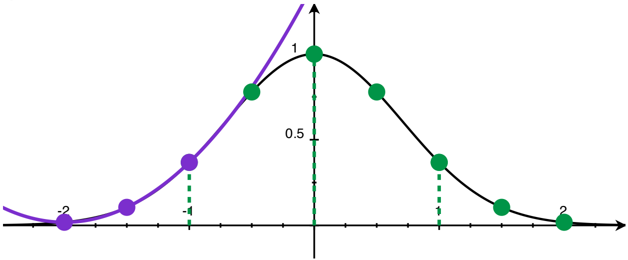

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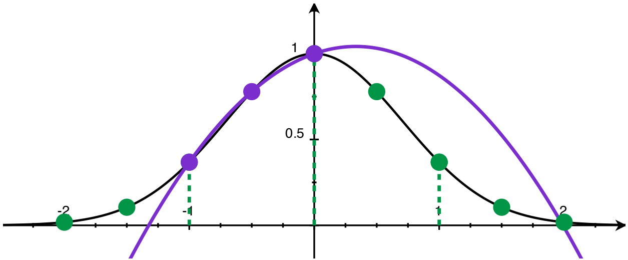

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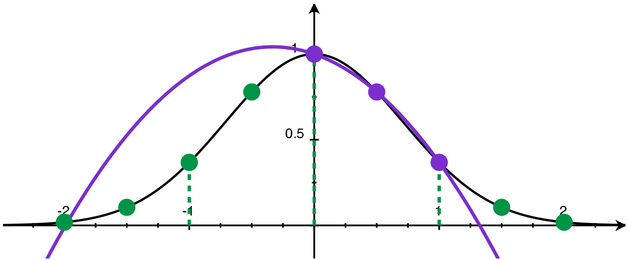

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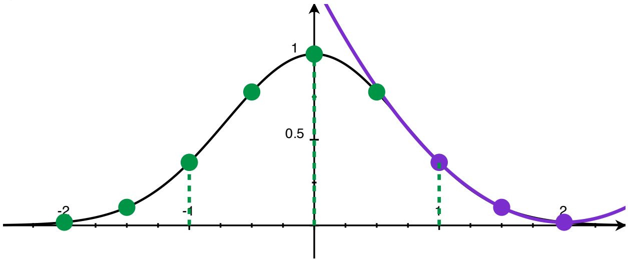

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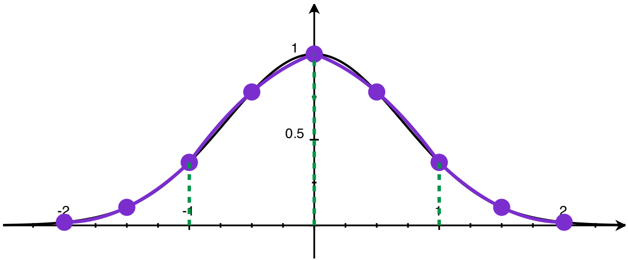

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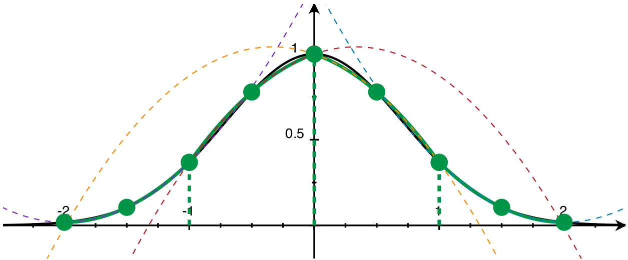

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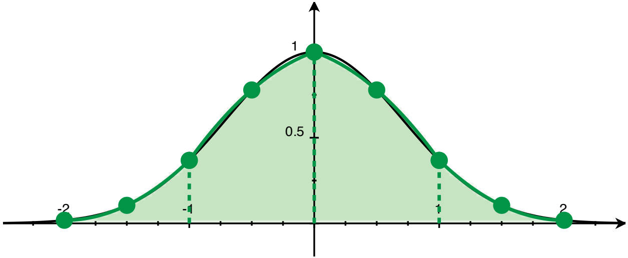

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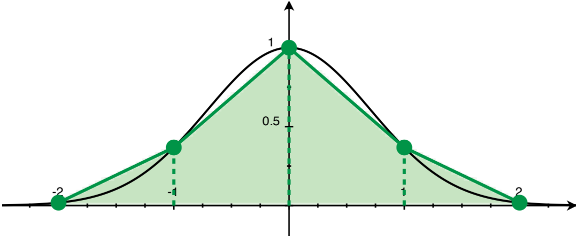

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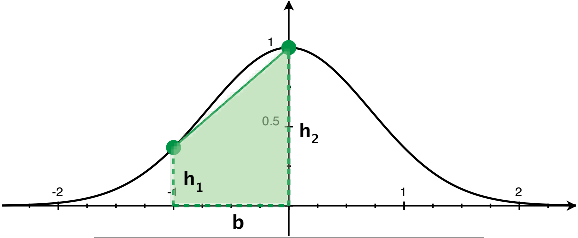

$$\boxed{\text{Area (trapeziod)} = b * \frac{h_1 + h_2}{2}}$$

For example: $b = 1, \quad h_1 = f(-1), \quad h_2 = f(0),$

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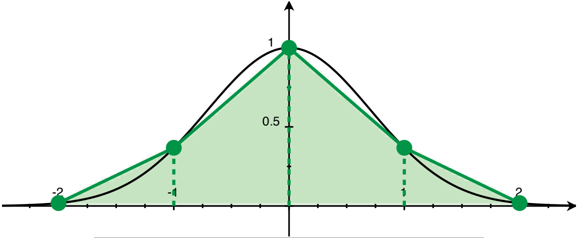

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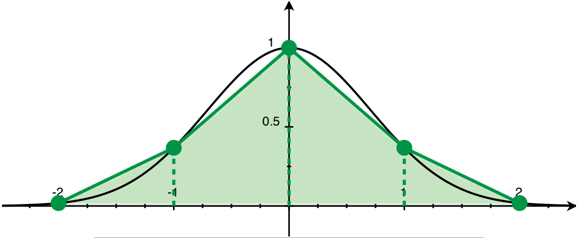

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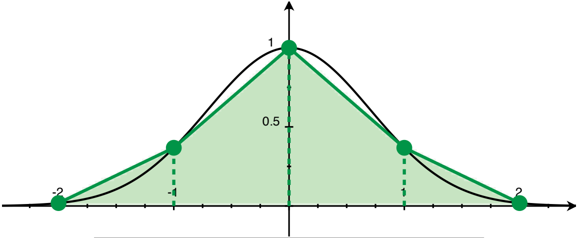

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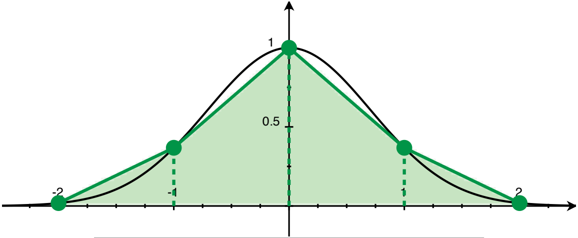

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(see book for general form)

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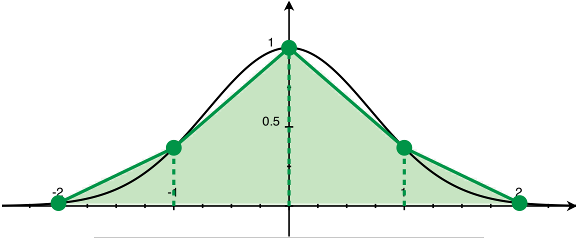

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- For **Simpson's rule** (parabolas), know how to use applet.

  Warning about conventions: In the book and webwork, n is the number of "subintervals". In class and in the applet, n is the number of parabolas. So if *webwork* says $n = 6$, plug in $n = 3$ to the *applet*.

Read about error!

M_4: maximum of $\left| \frac{d^4}{dx^4} f(x) \right|$

our $[a, b]$
total interval.