Antiderivatives and Initial Value Problems

Warm up

If $\frac{d}{dx}f(x) = 2x$, what is $f(x)$?

$$f(x) = x^2$$

Can you think of another function that $f(x)$ could be?

Some other candidates:

$$f(x) = x^2 + 1, \quad x^2 - 2, \quad x^2 + 13\pi, \quad x^2 - 143.7$$

If $\frac{d}{dx}f(x) = 3x^2 + 1$, what is $f(x)$?

$$f(x) = x^3 + x$$

Can you think of another function that $f(x)$ could be?

Some other candidates:

$$f(x) = x^3 + x + 1, \quad x^3 + x - 2, \quad x^3 + x + 13\pi, \quad x^3 + x - 143.7$$

Definition

An **antiderivative** of a function f on an interval I is another function F such that $F'(x) = f(x)$ for all $x \in I$.

Examples:

1. An antiderivative of $f(x) = 2x$ is $F(x) = x^2$.
2. Another antiderivative of $f(x) = 2x$ is $F(x) = x^2 + 1$.
3. There are *lots* of antiderivatives of $f(x) = 2x$ which look like $F(x) = x^2 + C$.

Suppose over $(a,b) = I$

* f is differentiable

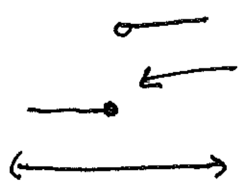
  defined over I

  Continuous over I

  no corners

* and $f' = 0$.

then f is constant.



not continuous.

$f = c$.

say $F(x)$ and $G(x)$ so that

$F'(x)$ and $G'(x)$ are both $2x$.

then
$$F'(x) - G'(x) = 2x - 2x = 0$$
$$\frac{d}{dx}\Big(\underbrace{F(x) - G(x)}_{\text{constant! } c}\Big) =$$

$$F(x) - G(x) = c \longrightarrow F(x) = G(x) + c.$$

Suppose that h is differentiable in an interval I,
and $h'(x) = 0$ for all x in I.

Then h is a constant function!
i.e. $h(x) = C$ for all $x \in I$, where C is a constant.

So, if $F(x)$ is one antiderivative of $f(x)$, then any other antiderivative must be of the form $F(x) + C$.

Example: **All** of the antiderivatives of $f(x) = 2x$ look like

$$F(x) = x^2 + C$$

for some constant C.

Every function f that has at least one antiderivative F has **infinitely many** antiderivatives

$$F(x) + C.$$

We refer to $F(x) + C$ as the

general antiderivative or the **indefinite integral**

and denote it by

$$F(x) + C = \int f(x) \mathrm{d}x.$$

Example:

$$\int 2x \; dx = x^2 + C.$$

Examples

$$\int x^2 \ dx = \tfrac{1}{3}x^3 + C, \qquad \text{because} \qquad \tfrac{d}{dx}(\tfrac{1}{3}x^3 + C) = \tfrac{1}{3} * 3x^2 = x^2$$

$$\int x^3 \ dx = \tfrac{1}{4}x^4 + C, \qquad \text{because} \qquad \tfrac{d}{dx}(\tfrac{1}{4}x^4 + C) = \tfrac{1}{4} * 4x^3 = x^3$$

the derivative

$$\frac{d}{dx} f$$

of f

versus x

$$\int f \, dx$$

verus x

of f

the antideriv

$$\frac{d}{dx} x^6 = 6x^5 \qquad \text{so} \qquad \frac{d}{dx} \frac{1}{6} x^6 = x^5$$

$$\text{so} \qquad \int x^5 \, dx = \frac{1}{6} x^6 + C .$$

$$\int x^{-3} \, dx = \frac{x^{-2}}{-2} + C \qquad \text{b/c} \qquad \frac{d}{dx} -\frac{1}{2} x^{-2} = x^{-3} .$$

There's no a s.t. $\frac{d}{dx} x^a = x^{-1}$.

Examples

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$$\int x^5\ dx = \tfrac{1}{6}x^6 + C, \qquad \text{because} \qquad \tfrac{d}{dx}(\tfrac{1}{6}x^6 + C) = \tfrac{1}{6} * 6x^5 = x^5$$

$$\int x^{-3}\ dx = \tfrac{1}{-2}x^{-2} + C, \qquad \text{because} \qquad \tfrac{d}{dx}(\tfrac{1}{-2}x^{-2} + C) = x^{-3}$$

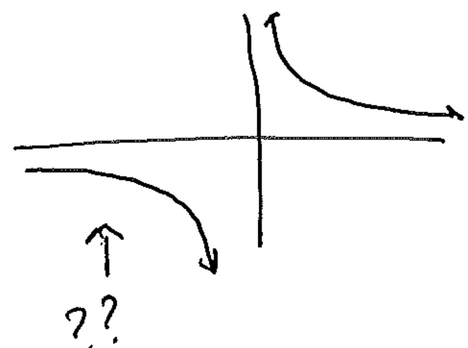
$$\int x^k\ dx = \tfrac{1}{k+1}x^{k+1} + C, \qquad \text{because} \qquad \tfrac{d}{dx}(\tfrac{1}{k+1}x^{k+1} + C) = x^k$$

Except!! What if $k = -1$?

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

so $\displaystyle\int \frac{1}{x}\,dx = \ln(x) + C$

domain: $x \neq 0$ — domain $x > 0$



Fix: for $x<0$,

$\frac{1}{x}$ is deriv of $\ln(-x)$

$$\int \frac{1}{x}\,dx = \begin{cases} \ln(x) & x>0 \\ \ln(-x) & x<0 \end{cases} = \underline{\underline{\ln|x| + C}}.$$

Some important basic integrals

$$\int x^k \, dx = \frac{1}{k+1}x^{k+1} + C \qquad \text{if } k \neq -1$$

$$\int x^{-1} \, dx = \ln|x| + C$$

$$\int \sin(x) \, dx = -\cos(x) + C$$

$$\int \cos(x) \, dx = \sin(x) + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \sec^2(x) \, dx = \tan(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x) + C = -\arccos(x) + D$$

$$\frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

<u>So</u> $\arcsin(x) + \arccos(x) = C$ for some C.

Theorem (Opposite of sum and constant rules)
Suppose the functions f and g both have antiderivatives on the interval I. Then for any constants a and b, the function $af + bg$ has an antiderivative on I and

$$\int (a * f(x) + b * g(x))dx = a\int f(x)dx + b\int g(x)dx$$

$$\frac{d}{dx} f*g = f'g + g'f \quad \text{so} \quad \int \underbrace{f'g + g'f}\, dx = fg.$$

Differential equations

A **differential equation** is an equation involving derivatives.
The *goal* is usually to solve for y.
Just like you could use algebra to solve

$$y^2 + x^2 = 1$$

for y, you can use calculus (and algebra) to solve things like

$$\frac{dy}{dx} - 5y = 0 \qquad \text{for } y.$$

A **solution** to a differential equation is a function you can plug in that satisfies the equation.
For example, $y = e^{5x}$ is a solution to the differential equation above since

$$\frac{d}{dx} e^{5x} = 5e^{5x},$$

so

$$\frac{dy}{dx} - 5y = (5e^{5x}) - 5(e^{5x}) = 0 \quad \checkmark$$

Simplest differential equations: antiderivatives

Finding an antiderivative can also be thought of as solving a differential equation:

"Solve the differential equation $\frac{d}{dx}y = x^2$."

Answer: $$y = \int x^2 dx = \frac{1}{3}x^3 + C.$$

Check: $$\frac{d}{dx}\frac{1}{3}x^3 + C. = \frac{1}{3} * 3 * x^2 + 0 = x^2 \quad \checkmark$$

Examples

(1) Solve the differential equation $y' = 2x + \sin(x)$.

$$y = x^2 - \cos(x) + C$$

(2) Check that $\cos(x) + \sin(x)$ is a solution to $\dfrac{d^2y}{dx^2} + y = 0$.

$$\frac{d}{dx}y = \frac{d}{dx}(\cos(x) + \sin(x)) = -\sin(x) + \cos(x), \quad \text{so}$$

$$\frac{d^2}{dx^2}y = -\cos(x) - \sin(x) = -(\cos(x) + \sin(x)).$$

Therefore, $\dfrac{d^2y}{dx^2} + y = -(\cos(x) + \sin(x)) + (\cos(x) + \sin(x)) = 0 \quad \checkmark$

you can check that

$$y = A\cos(x) + B\sin(x)$$

for <u>any</u> constants A & B

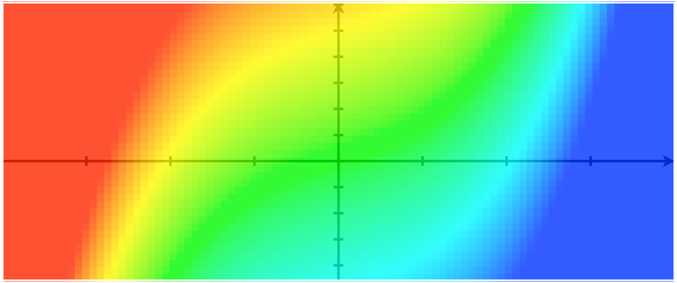
will be solns to

$$y'' + y = 0.$$

Initial value problems

Find a solution to the differential equation $\frac{d}{dx}y = x^2 + 1$ which *also* satisfies $y(2) = 8/3$.

$$\boxed{\text{general solution:} \quad y = \frac{1}{3}x^3 + x + C}$$



Each color corresponds to a choice of C.

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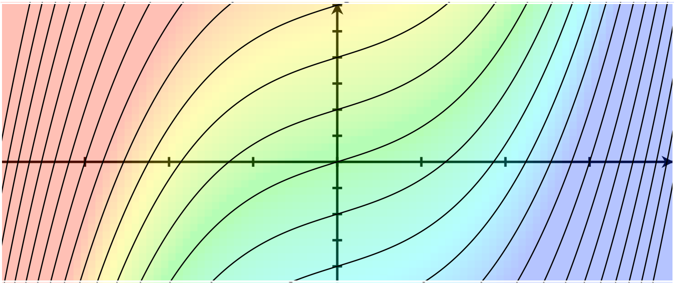

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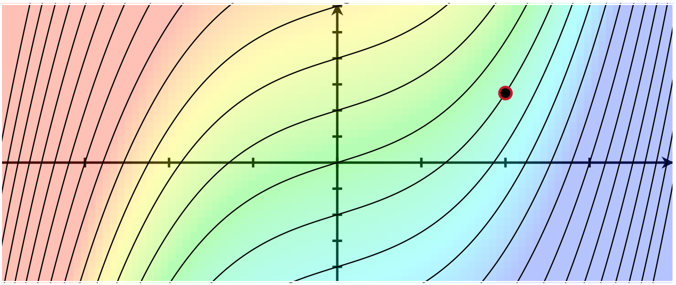

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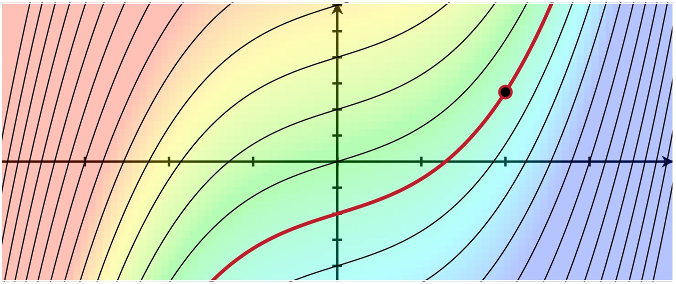

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Red cuve is the *particular* solution.

$$y = \frac{1}{3}x^3 + x + c$$

$$\frac{8}{3} = y(2) = \frac{1}{3}(2)^3 + 2 + c$$

solve for c:

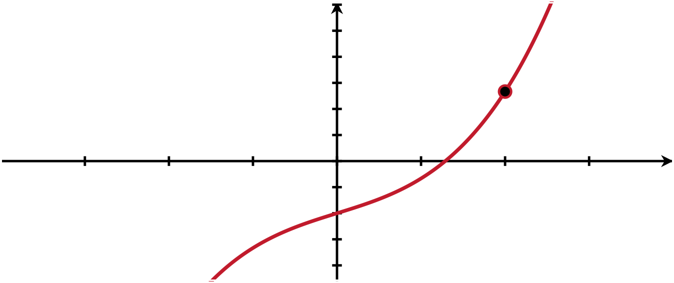
$$\frac{8}{3} - \frac{1}{3}\cdot 8 - 2 = c$$

$$\Rightarrow \boxed{c = -2}$$

Initial value problems

Find a solution to the differential equation $\frac{d}{dx}y = x^2 + 1$ which *also* satisfies $y(2) = 8/3$.

general solution:	$y = \frac{1}{3}x^3 + x + C$
particular solution:	$y = \frac{1}{3}x^3 + x - 2$



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Red cuve is the *particular* solution.

Definition

An **initial-value problem** is a differential equation together with enough additional conditions to specify the constants of integration that appear in the general solution.

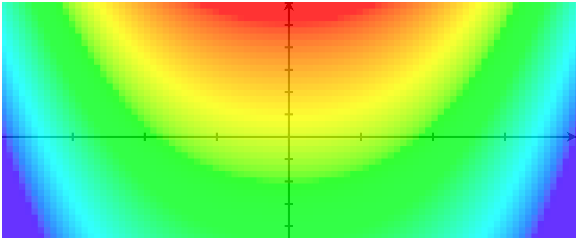
The **particular solution of the problem** is then a function that satisfies both the differential equation and also the additional conditions.

Solve the initial value problem

$$\frac{dy}{dx} = 2x + \sin(x)$$

subject to $y(0) = 0$.

$$\boxed{\text{general solution:} \quad y = x^2 - \cos(x) + C}$$



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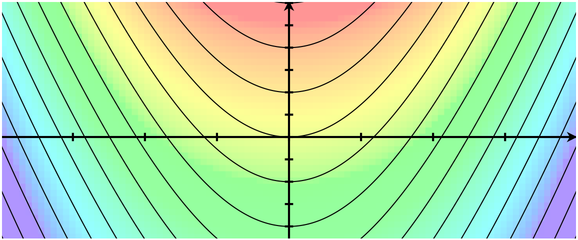

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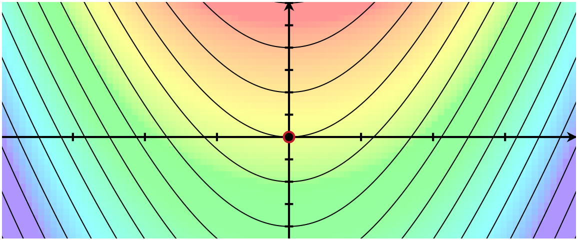

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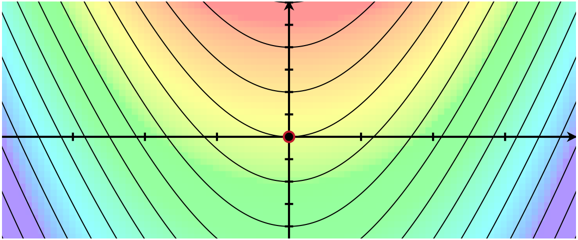

Algebraically: get a particular solution by solving

$$\mathbf{0} = \mathbf{y(0)} = (0)^2 - \cos(0) + C = -1 + C \quad (\text{for } C)$$

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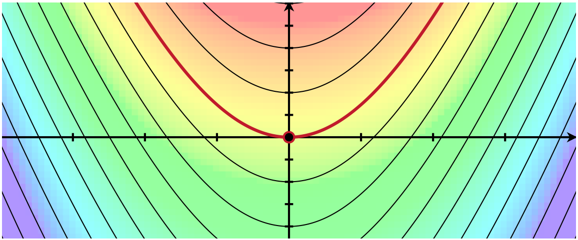

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$$\mathbf{0} = \mathbf{y(0)} = (0)^2 - \cos(0) + C = -1 + C \quad (\text{for } C)$$

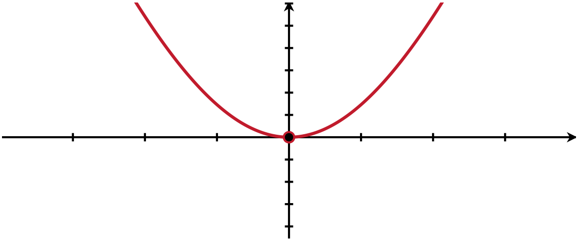
$$C = 1, \qquad \text{so } \boxed{y = x^2 - \cos(x) + 1.}$$

Solve the initial value problem

$$\frac{dy}{dx} = 2x + \sin(x)$$

subject to $y(0) = 0$.

$$\boxed{\text{general solution:} \quad y = x^2 - \cos(x) + C}$$



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$$C = 1, \qquad \text{so } \boxed{y = x^2 - \cos(x) + 1.}$$

$\boxed{\text{Solve the initial-value problem} \quad y'' = \cos x,\ y'(\frac{\pi}{2}) = 2,\ y(\frac{\pi}{2}) = 3\pi.}$

Step 1: Calculate the antiderivative of $\cos(x)$ to find the general solution for y'. Ans: $\boxed{y' = \sin(x) + C}$

Step 2: Plug in the values $y'(\frac{\pi}{2}) = 2$ to calculate C.
Ans: $2 = \sin(\pi/2) + C = 1 + C$, so $\boxed{C = 1}$

Step 3: Write down the *particular* solution for y'. Ans: $\boxed{y' = \sin(x) + 1}$

Step 4: Calculate the antiderivative of your particular solution in Step 3 to find the *general solution for* y.
Ans: $\boxed{y = -\cos(x) + x + D}$

Step 5: Plug in the values $y(\frac{\pi}{2}) = 3\pi$ to solve for the new constant.
Ans: $3\pi = -\cos(\pi/2) + \pi/2 + D = \pi/2 + D$ so $\boxed{D = 5\pi/2}$

Step 6: Write down the *particular* solution for y.
Ans: $\boxed{y = -\cos(x) + x + 5\pi/2}$

$$y'' = \cos(x)$$

so $\quad y' = \int \cos(x)\,dx$

$$= \sin(x) + C$$

so $\quad y = \int \sin(x) + C \quad dx$

$$= -\cos(x) + C * x + D.$$

general soln.

Word problem:

An object dropped from a cliff has acceleration $a = -9.8\ m/sec^2$ under the influence of gravity. What is the function $s(t)$ that models its height at time t?

Initial value problem:

Solve

$$\frac{d^2s}{dt^2} = -9.8, \qquad s(0) = s_0,\ s'(0) = 0.$$

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$$s' = \int -9.8\,dt = -9.8t + c$$

$$0 = s'(0) = -9.8 \cdot 0 + c \qquad \text{so} \qquad \boxed{c = 0}$$

$$\text{so} \qquad \boxed{s' = -9.8t}$$

Then

$$s = \int s'\,dt = \int -9.8t\,dt = -9.8 \cdot \tfrac{1}{2}t^2 + D$$
$$= -4.9t^2 + D$$

$$s_0 = s(0) = 0 + D \qquad \text{so} \qquad D = s_0$$

$$\text{so} \qquad \boxed{s(t) = -4.9t^2 + s_0}$$

Word problem:
Suppose that a baseball is thrown upward from the roof of a 100 meter high building. It hits the street below eight seconds later. What was the initial velocity of the baseball, and how high did it rise above the street before beginning its descent?

Initial value problem:
Solve
$$\frac{d^2s}{dt^2} = -9.8, \qquad s(0) = 100,\ s(8) = 0.$$
Use your solution to
(1) calculate $s'(0)$, and
(2) solve $s'(t_1) = 0$ for t_1 and calculate $s(t_1)$.

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Use your solution to
(1) calculate $s'(0)$, and
(2) solve $s'(t_1) = 0$ for t_1 and calculate $s(t_1)$.

$$s' = \int -9.8\, dt = \boxed{-9.8t + c} \quad s'$$

so $s = \int s'\,dt = \int -9.8t + c\ dt$

$$= -9.8 \cdot \tfrac{1}{2}t^2 + C \cdot t + D$$

$$\boxed{s(t) = -4.9t^2 + C \cdot t + D}$$

$100 = s(0) = 0 + 0 + D \longrightarrow \boxed{D = 100}$

$0 = s(8) = -4.9(8)^2 + C \cdot 8 + 100$

so $C = \dfrac{4.9 \cdot 8^2 - 100}{8} = \boxed{4.9 * 8 - 12.5}$

26.7

(1) $s'(0) = C = 4.9 * 8 - 12.5 = 26.7$

(2) $0 = s'(t_1) = -9.8 \cdot t_1 + 26.7 \rightarrow \boxed{t_1 = \dfrac{-26.7}{-9.8}} = +2.724$

$s(t_1) = -4.9(+2.724)^2 + (26.7)(+2.724) + 100$