Optimization

Warm up

Sketch the graph of

$$f(x) = (x-3)(x-2)(x-1) = x^3 - 6x^2 + 11x - 6$$

over the interval $[1, 4]$. Mark any critical points and inflection points. What is the absolute maximum over this interval? What is the absolute minimum over this interval?
[useful value: $\sqrt{3}/3 \approx .6$]

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Sketch the graph of

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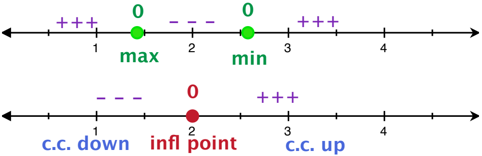
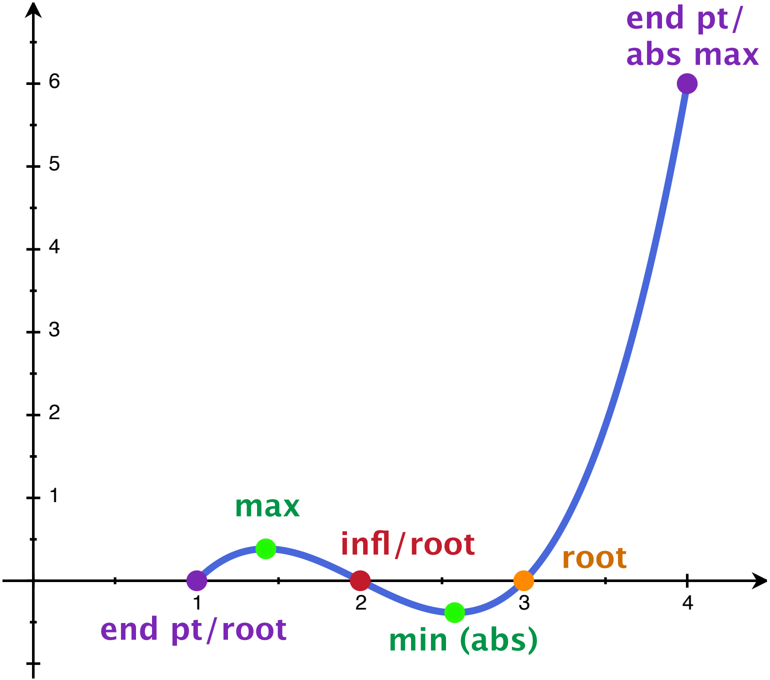
[useful value: $\sqrt{3}/3 \approx .6$]

$$f'(x) = 3x^2 - 12x + 11 = 3\left(x - (2 - \sqrt{3}/3)\right)\left(x - (2 + \sqrt{3}/3)\right)$$

$$f''(x) = 6x - 12 = 6(x-2)$$

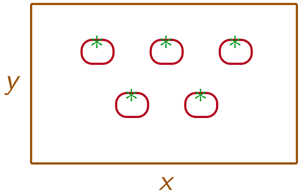


$f(2 - \sqrt{3}/3) \approx 0.385$
$f(2 + \sqrt{3}/3) \approx -0.385$
$f(1) = 0$
$f(4) = 6$



end pt/
abs max
6
5
4
3
2
1
max
infl/root
root
1
2
3
4
end pt/root
min (abs)

Suppose you want to fence off a garden, and you have 100m of fence. What is the largest area that you can fence off?



Get it into math:

Know: $2x + 2y = 100$ Want: Maximize $A = xy$

Problem: The area, xy, is a function of two variables!!
Strategy: Use the first equation to get xy into one variable: Solve $2x + 2y = 100$ (the "constraint") and plug into xy (the function you want to optimize).

$$2x + 2y = 100 \quad \Longrightarrow \quad y = 50 - x$$

so $xy = x(50 - x) = 50x - x^2$.

$\boxed{\text{Domain: } 0 \leq x \leq 50}$

New problem: Maximize $A(x) = 50x - x^2$ over the interval $0 < x < 50$.

Solution...

Three strategies:

(1) First derivative test:

(2) Pretend we're on a closed interval, then throw out the endpoints:

(3) Second derivative test:

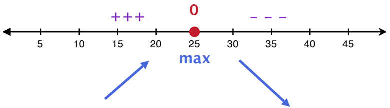
$\boxed{\text{New problem: Maximize } A(x) = 50x - x^2 \text{ over the interval } 0 < x < 50.}$

Solution: $A'(x) = 50 - 2x$

So the only critical point is when $50 - 2x = 0$, so $\boxed{x = 25}$.

Three strategies:

(1) First derivative test:



$\boxed{\text{So } A(25) = 625 \text{ is maximal.}}$

(2) Pretend we're on a closed interval, then throw out the endpoints:

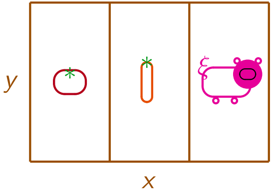
x	$A(x)$	
25	$\boxed{625}$	max!
0	0	min
50	0	min

Since the maximum is not at one of the points I have to throw out, it must me a maximum on the open interval (there is no absolute minimum over the open interval $(0, 50)$).

(3) Second derivative test:

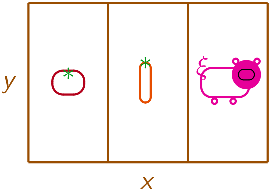
$$A''(x) = -2 < 0 \text{ so } A(25) = 625 \text{ must be a maximum.}$$

Now suppose, instead, you want to divide your plot up into three equal parts:



If you still only have 100 m of fence, what is the largest area that you can fence off?

Now suppose, instead, you want to divide your plot up into three equal parts:



If you still only have 100 m of fence, what is the largest area that you can fence off?

Solution:

Constraint: $2x + 4y = 100$, so $y = 25 - \frac{1}{2}x$.

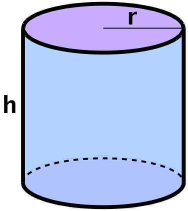
Maximize: $A = xy$ over $0 < x < 50$.

Plug in constraint: $A(x) = x(25 - x/2) = 25x - x^2/2$

Find critical points: $0 = A'(x) = 25 - x$, so $x = 25$.

Second derivative test: $A''(x) = -1 < 0$ so $\boxed{A(25) = 25 * 12.5}$ is a maximum.

Suppose you want to make a can which holds about 16 ounces (28.875 in^3). If the material for the top and bottom of the can costs 4 ¢/in^2 and the material for the sides of the can costs 3 ¢/in^2. What is the minimum cost for the can?



Put into math:
Constraint: $V = \pi r^2 h = 28.875.$
Cost: $4 *$ (SA of top + bottom) $+3 *$ (SA of side)

Top: πr^2 Bottom: πr^2 Sides: $(2\pi r)h$

$$\boxed{\text{Total cost: } C = 4 * 2 * (\pi r^2) + 3 * ((2\pi r)h)}$$

Get into one variable: Use the constraint!

$$\pi r^2 h = 28.875 \implies h = \frac{28.875}{\pi} r^{-2} \implies C(r) = 8\pi r^2 + 6\pi r \left(\frac{28.875}{\pi} r^{-2}\right)$$

So

$$\boxed{C(r) = 8\pi r^2 + 6 * 28.875 r^{-1}}$$

(Domain: $r > 0$)

$$\boxed{\text{New problem: Minimize } C(r) = 8\pi r^2 + 6 * 28.875 r^{-1} \text{ for } r > 0.}$$

[hint: If you don't have a calculator, use the second derivative test!]

New problem: Minimize $C(r) = 8\pi r^2 + 6 * 28.875 r^{-1}$ for $r > 0$.

[hint: If you don't have a calculator, use the second derivative test!]

Solution: $(6 * 28.875 = 173.25)$

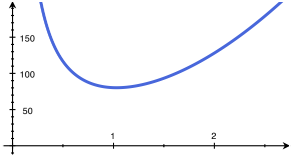
$$C'(r) = 16\pi r - 173.25 r^{-2} = \frac{1}{r^2}\left(16\pi r^3 - 173.25\right)$$

Critical point: $C'(r) = \sqrt[3]{\frac{173.25}{16\pi}} \approx 1.031$

Second derivative test: $C''(r) = 16\pi + 173.25 r^{-3} > 0$ when $r > 0$,

so $C(r)$ is concave up, and so $C\left(\sqrt[3]{\frac{173.25}{16\pi}}\right)$ is a minimum.

Minimal value: $C\left(\sqrt[3]{\frac{173.25}{16\pi}}\right) \approx 80.2041¢$



Extra practice: Optimization

1. Find the local maxima and minima of $f(x) = x^4 - 62x^2 + 120x + 9$.
2. Find the local maxima and minima of $f(x) = (x-1)(x+2)^2$.
3. Show that $f(x) = x + 1/x$ has a local maximum and a local minimum, but the value at the local maximum is less than the value at the local minimum.
4. Find the absolute maxima and minima of $f(x) = -(x-1)^3(x+1)^2$ on the interval [-2,1.5].
5. Find the absolute maxima and minima of $f(x) = x/2 + 2/x$ for x in $[3, 7]$.
6. Find the absolute maxima and minima of $f(x) = 2x^3 - 24x + 107$ in the interval $[1, 3]$.
7. Find the absolute maxima and minima of $f(x) = \sin x + (1/2)\cos x$ in $0 \leq x \leq \pi/2$.
8. Find the maximum profit that a company can make if the profit function is given by $p(x) = 41 + 24x - 18x^2$.
9. An enemy jet is flying along the curve $y = x^2 + 2$. A soldier is placed at the point $(3, 2)$. At what point will the jet be at when the soldier and the jet are closest?
10. Find the local maxima and minima of $f(x) = -x + 2\sin x$ in $[0, 2\pi]$.
11. Divide 15 into two parts such that the square of one times the cube of the other is maximum.
12. Suppose the sum of two numbers is fixed. Show that their product is maximum exactly when each one of them is half of the total sum.
13. Divide a into two parts such that the pth power of one times the qth power of the other is maximum.
14. Which fraction exceeds its pth power by the maximum amount?
15. Find the dimensions of the rectangle of area 96 cm^2 which has minimum perimeter. What is this minimum perimeter?
16. Show that the right circular cone with a given volume and minimum curved surface area has altitude equal to $\sqrt{2}$ times the radius of the base.
17. Show that the altitude of the right circular cone with maximum volume that can be inscribed in a sphere of radius R is $4R/3$.
18. Show that the height of a right circular cylinder with maximum volume that can be inscribed in a given right circular cone of height h is $h/3$.

1. Local extrema of

$$f(x) = x^4 - 62x^2 + 120x + 9$$

$$f'(x) = 4x^3 - 124x + 120$$

$$= 4(x^3 - 31x + 30) = 4(x-1)(x^2+x-30)$$

Factoring: $f'(1) = 0$:

$$\begin{array}{r} x^2 + x - 30 \\ x - 1 \overline{\big)\, x^3 - 31x + 30} \\ -(x^3 - x^2) \\ \hline x^2 - 31x + 30 \\ -(x^2 - x) \\ \hline -30x + 30 \end{array}$$

$$= \boxed{4(x-1)(x+6)(x-5)}$$

f' : $-$ 0 $+$ 0 $-$ 0 $+$

$-6 \qquad 1 \qquad 5$

$\searrow \quad \nearrow \quad \searrow \quad \nearrow$

local mins : $\boxed{x = -6, 5}$

local max's : $\boxed{x = 1}$

2. Local extrema of

$$f(x) = (x-1)(x+2)^2.$$

To make factoring easier, I don't expand first...

$$f' = (x+2)^2 + 2(x-1)(x+2)$$

$$= (x+2)(x+2+2x-2)$$

$$= (x+2)(3x)$$

f' : $+$ $-$ $+$

$-2 \qquad 0$

$\nearrow \quad \searrow \quad \nearrow$

local max : $\boxed{x = -2}$

local min : $\boxed{x = 0}$

19. A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions of the can which will minimize the cost of the metal to make the can.

20. An open box is to be made out of a given quantity of cardboard of area p^2. Find the maximum volume of the box if its base is square.

21. Show that $f(x) = \sin x(1 + \cos x)$ is maximum when $x = \pi/3$.

22. An 8 inch piece of wire is to be cut into two pieces. Figure out where to cut the wire in order to make the sum of the squares of the lengths of the two pieces as small as possible.

23. Find the dimensions of the maximum rectangular area that can be fenced with a fence 300 yards long.

24. Given the perimeter of a rectangle show that its diagonal is minimum when it is a square. Make up a word problem for which this gives the solution.

25. Prove that the rectangle of maximum area that can be inscribed in a circle is a square. Make up a word problem for which this gives the solution.

26. Show that the triangle of the greatest area with given base and vertical angle is isosceles.

27. Show that a right triangle with a given perimeter has greatest area when it is isosceles.

28. Show that the angle of the cone with a given slant height and with maximum volume is $\tan^{-1}(\sqrt{2})$.

1. Local extrema of

$$f(x) = x^4 - 62x^2 + 120x + 9$$

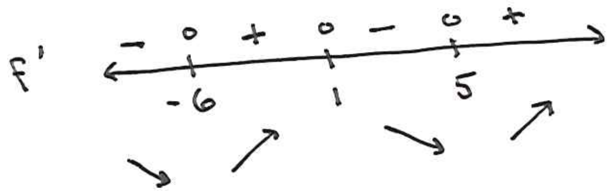
$$f'(x) = 4x^3 - 124x + 120$$

$$= 4(x^3 - 31x + 30) = 4(x-1)(\underbrace{x^2+x-30})$$

Factoring: $f'(1) = 0$:

$$\begin{array}{r} x^2 + x - 30 \\ x-1 \,\big)\overline{\, x^3 - 31x + 30} \\ -(x^3 - x^2) \\ \hline x^2 - 31x + 30 \\ -(x^2 - x) \\ \hline -30x + 30 \end{array}$$

$$= \boxed{4(x-1)(x+6)(x-5)}$$



local mins : $\boxed{x = -6, 5}$

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2. Local extrema of

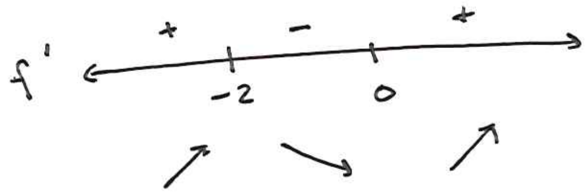
$$g(x) = (x-1)(x+2)^2.$$

To make factoring easier, I don't expand first...

$$f' = (x+2)^2 + 2(x-1)(x+2)$$

$$= (x+2)(x+2+2x-2)$$

$$= (x+2)(3x)$$



local max : $\boxed{x = -2}$

local min : $\boxed{x = 0}$

— Setting some up —

9. constraint: $y = x^2 + 2$

minimize distance

$$d = \sqrt{(x-3)^2 + (y-2)^2}$$

10. ✓

11. Constraint: $x + y = 15$

maximize $x^2 y^3$

12. constr. $x + y = a$

max: xy

13. $a = x + y$ ← constr.

$x^p + y^q$ ← maximize

14. Maximize $x - x^p$ (p is some #)

15. constr: $xy = 96$

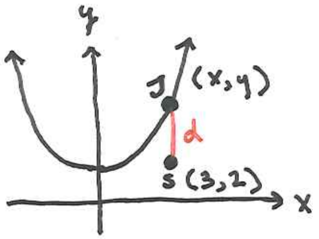
minimize $2x + 2y$.

16. constr: $\dfrac{\pi r^2 h}{3} = c$ (for some c)

minimize surface area: $\pi r \sqrt{r^2 + h^2}$

$s = \sqrt{r^2 + h^2}$ ("altitude" means height)

9. An enemy jet is flying along the curve $y = x^2+2$. A soldier is placed at the point $(3,2)$. At what point will the jet be when it's closest to the soldier?



optimize: $d = \sqrt{(x-3)^2+(y-2)^2}$

constraint: $y = x^2+2$

plug in

$$d(x) = \sqrt{(x-3)^2 + \underbrace{(x^2+2-2)^2}_{x^4}} = \sqrt{x^4+x^2-6x+9}$$

$$\text{so } d'(x) = \frac{\cancel{4}^{2}x^3 + \cancel{2}x - \cancel{6}^{3}}{\cancel{2}\sqrt{x^4+x^2-6x+9}} = \frac{2x^3+x-6}{\sqrt{x^4+(x-3)^2}}$$

denom: $x^4+(x-3)^2$ always positive

num: $2x^3+x-3 = 0$ @ $x=1$ (only)

d': $-$ 0 $+$ at $x = 1$

$\searrow$ min $\nearrow$

local min is only turn $\to$ absolute min

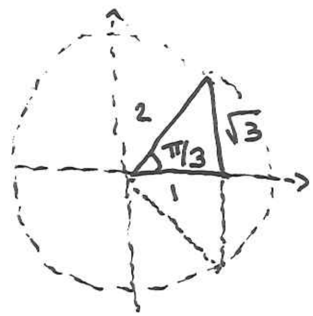
$\Rightarrow$ $\boxed{(1,3)}$

10. Find local extrema of $f = -x+2\sin(x)$ in $[0, 2\pi]$.

$f' = -1+2\cos(x)$

find crit pts: $-1+2\cos(x) = 0$

so $\cos(x) = 1/2$



In the interval $[0, 2\pi]$:

$x = \pi/3,\ \underbrace{2\pi - \frac{\pi}{3}}_{5\pi/3}$

f': e.p. [0 $(+)$ $\pi/3$ (0) $(-)$ $5\pi/3$ (0) $(+)$ 2π] e.p.

$\nearrow$ $\searrow$ $\nearrow$

local mins: $\boxed{x = 0, 5\pi/3}$

local max's: $\boxed{x = \pi/3, 2\pi}$

11. Divide 15 into two parts so that the square of one times the cube of the other is biggest.

Constraint: $x+y=15$, $15 \geq x, y \geq 0$.

Maximize: $P = x^2 \cdot y^3$.

$x+y=15 \rightarrow y = 15-x$ or $\underline{\underline{x = 15-y}}$.

(since square is smaller, or whatever)

so

$$P = (15-y)^2 \cdot y^3$$

$$\frac{d}{dy}P = -2(15-y)y^3 + 3(15-y)^2y^2$$

$$= (15-y)\underbrace{\left(-2y^3 + 3(15-y)y^2\right)}_{45y^2-(2+3)y^3}$$

$$= 5(15-y)y^2(9-y)$$

y	P
0	0
9	$\boxed{6^2 \cdot 9^3}$
15	0

$\rightarrow$ max P @ $x=6$, $y=9$

12. Suppose the sum of two numbers is fixed. Show their product is maximized when one is half the total sum.

Constraint: $x+y=S$ $\quad S \geq x, y \geq 0$

maximize $P = xy$

$$= (S-y)y = Sy - y^2$$

$$\frac{dP}{dy} = S - 2y = 0 \quad \text{when } y = \frac{S}{2}$$

	0		$S/2$		S
	ep	+	0	−	ep

min ↗ max ↘ min

-or-

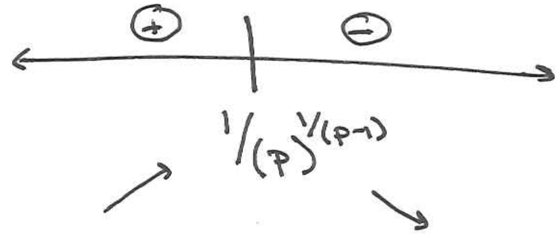
y	P
0	0
$S/2$	$S^2/4$ ← max.
S	0

14. Which fraction exceeds its p^{th} power by the maximum amount?

maximize: $f(x) = x - x^p$ (assume $p > 0$)

$f' = 1 - px^{p-1} = 0$ if $x^{p-1} = \frac{1}{p}$

$x = \frac{1}{(p)^{1/(p-1)}}$

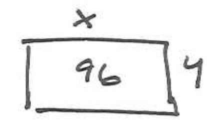


$$\boxed{x = \frac{1}{p^{1/p-1}}}$$

15. Find the dimensions of the rectangle of area 96 cm² which has min perimeter. What is that perimeter?

constraint: $xy = 96$, $x, y \geq 0$.

optimize: $2x + 2y = P$



$xy = 96 \rightarrow y = 96 \cdot \frac{1}{x}$

so

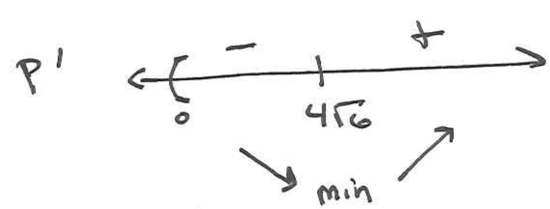
$P = 2\left(x + 96 \cdot \frac{1}{x}\right)$

so $P' = 2\left(1 - 96 \cdot \frac{1}{x^2}\right) = 0$

when $x^2 = 96$

$x = \pm 4\sqrt{6}$

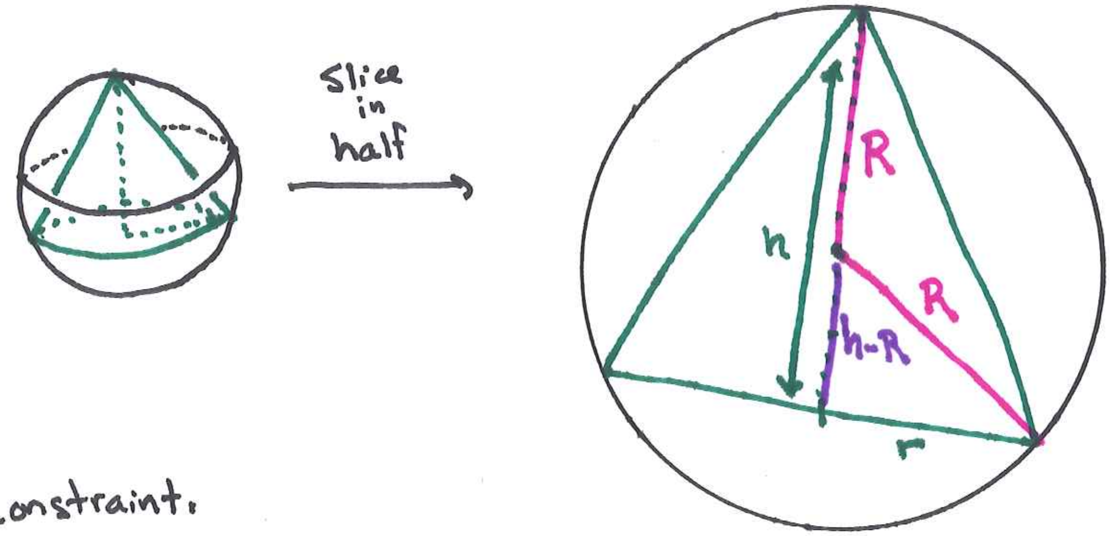
pos since $x \geq 0$.



only turn around ⇒ absolute min.

$$\boxed{\text{dimensions: } 4\sqrt{6} \times 4\sqrt{6}, \quad \text{perim: } 16\sqrt{6}}$$

17.



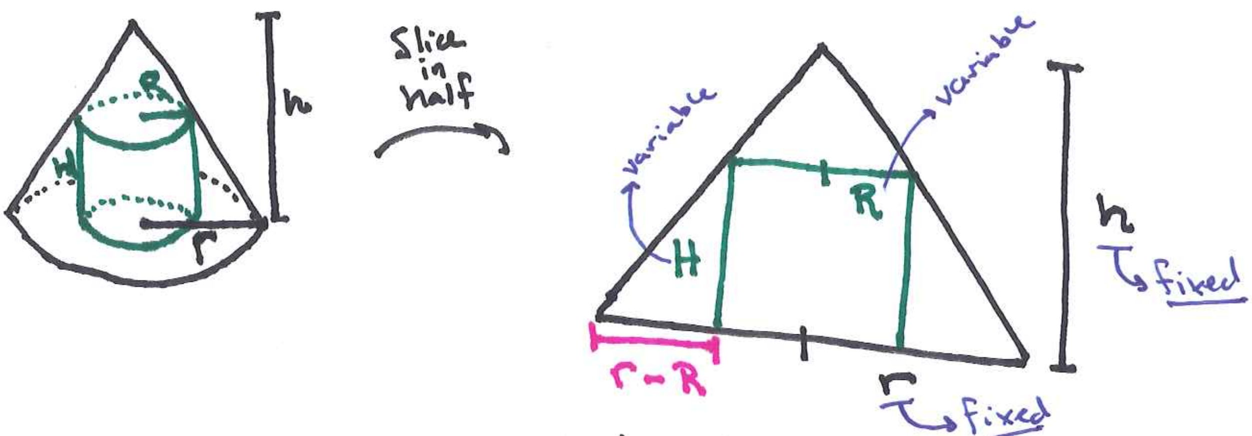
Constraint:

$(h-R)^2 + r^2 = R^2$ (R is a fixed number)

maximize

Volume $= \frac{1}{3}\pi r^2 h$ (in terms of h)

18.



similar triangles:

$$\frac{H}{r-R} = \frac{h}{r}$$ ← constraint

(h and r are constants)

maximize

$V = \pi R^2 H$ as a function of H

(taking it a little further)

constraint says $r - R = \frac{Hr}{h}$, so $R = r\left(1 - \frac{H}{h}\right)$.

Plug in to volume:

$$V(H) = \pi r^2 \left(1 - \frac{1}{h}H\right)^2 \cdot H$$

constant numbers variable

(show $H = h/3$ gives max)

★ won't depend on r!! (volume does, but critical H-value doesn't)