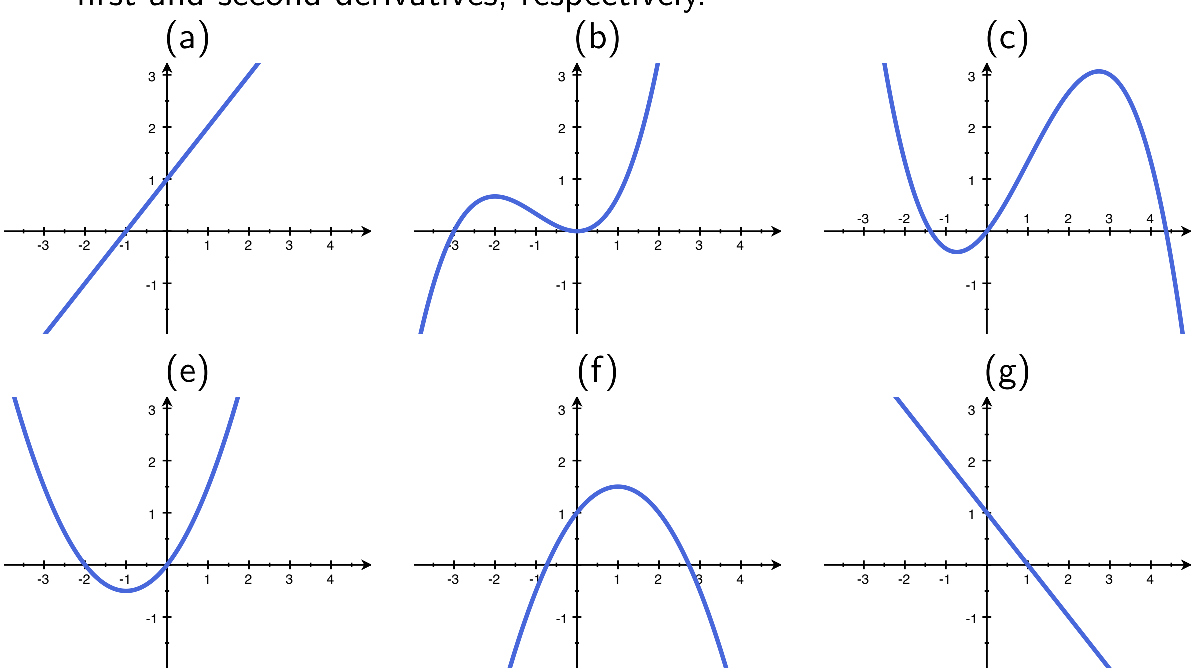
Curve Sketching

Warm up

Below are pictured six functions: f, f', f'', g, g', and g''. Pick out the two functions that could be f and g, and match them to their first and second derivatives, respectively.

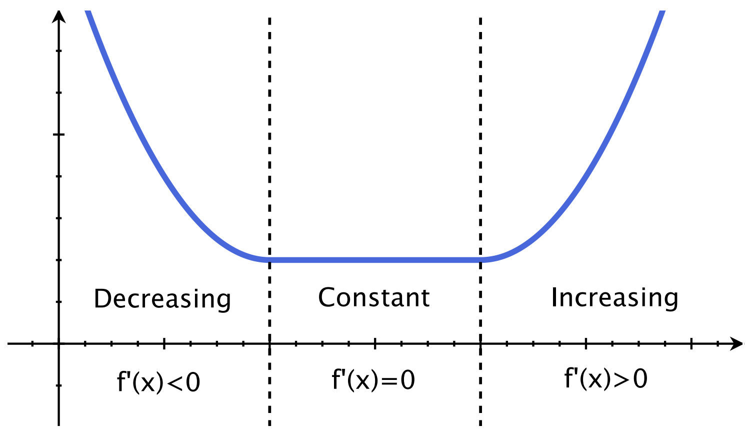


Review: Increasing/Decreasing

Suppose that f is continuous on $[a, b]$ and differentiable on the open interval (a, b) . Then

$$\text{If } f'(x) \text{ is } \left\{\begin{matrix}\text{positive}\\ \text{negative}\\ \text{zero}\end{matrix}\right\} \text{ for every } x \text{ in } (a, b) \text{ then } f \text{ is } \left\{\begin{matrix}\text{increasing}\\ \text{decreasing}\\ \text{constant}\end{matrix}\right\} \text{ on } [a, b].$$

What it looks like:

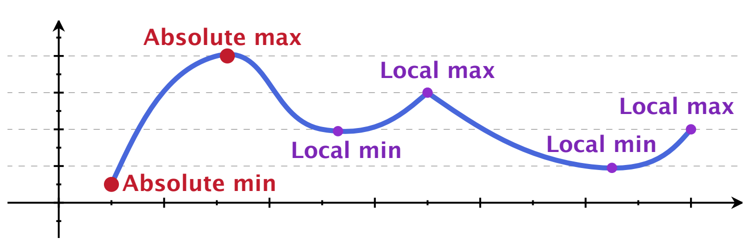


Review: Extreme values

If f is continuous on a closed interval $[a, b]$, then there is a point in the interval where f is largest (**maximized**) and a point where f is smallest (**minimized**).

The maxima or minima will happen either

1. at an endpoint, or
2. at a **critical point**, a point c where $f'(c) = 0$ or $f'(c)$ is undefined.
   Vocab: If $f'(c)$ is undefined, c is also called a **singular point**.



Review: finding **absolute** min/max on **closed** intervals

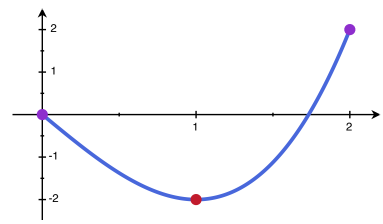
1. Calculate $f'(x)$.
2. Find where $f'(x)$ is 0 or undefined on $[a, b]$ (critical/singular points).
3. Evaluate $f(x)$ at the critical and singular points, and at endpoints. The largest (reps. smallest) value among these is the maximum (reps. minimum).

Example: Let $f(x) = x^3 - 3x$. What are the min/max values on the interval $[0, 2]$.

$$f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$$

So $f'(x) = 0$ if $x = -1$ or $\boxed{1}$.

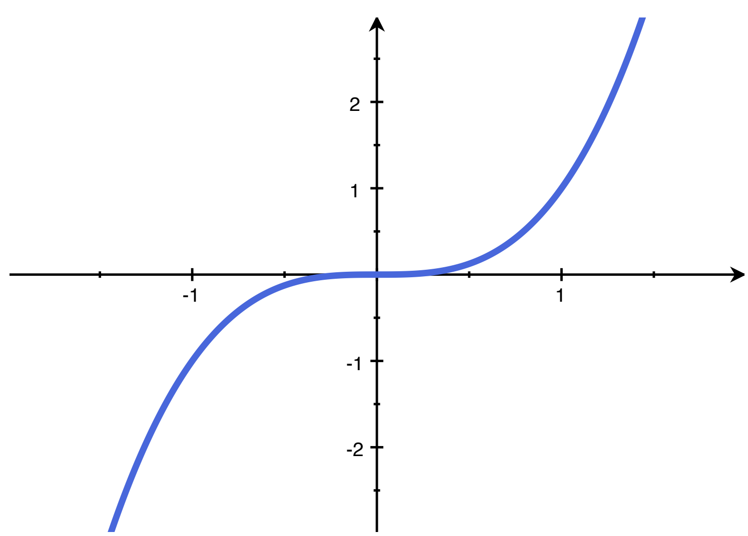
x	$f(x)$	
1	-2	critical points
0	0	end points
2	2	



Finding **local** min/max on **any** intervals

Warning: Not all critical points are local minima or maxima:

Example: If $f(x) = x^3$, then $f'(x) = 3x^2$, and so $f'(0) = 0$:



Finding local extrema: The First Derivative Test

Suppose

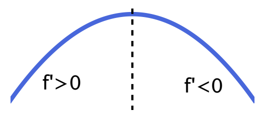
f is **continuous** on (a, b),

c is in (a, b) and is a **critical point** of $f(x)$, and

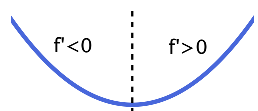
f is **differentiable** on (a, b) (except possibly at $x = c$)

Then the value $f(c)$ can be classified as follows:

1. If $f'(x)$ changes from **positive** $\to$ **negative** at $x = c$, then $f(c)$ is a **local maximum**.



2. If $f'(x)$ changes from **negative** $\to$ **positive** at $x = c$, then $f(c)$ is a **local minimum**.



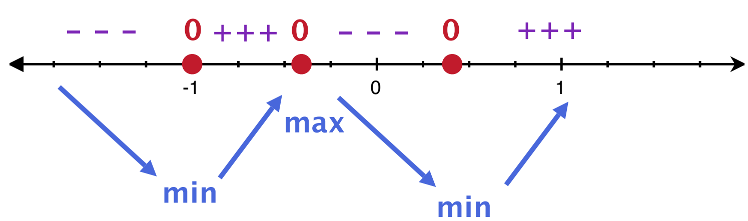
3. If $f'(x)$ doesn't change sign, then it's neither a min or a max.

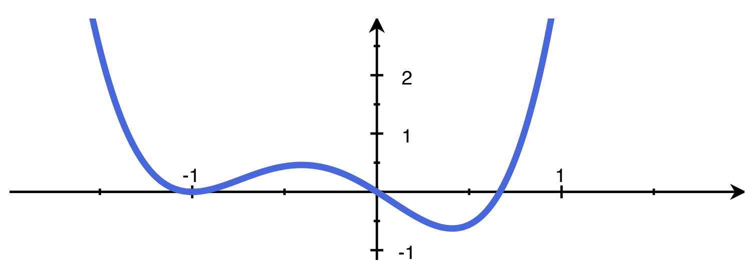
Example

Find the local extrema of $f(x) = 3x^4 + 4x^3 - x^2 - 2x$ over the whole real line.

Calculate $f'(x)$:

$$f'(x) = 12x^3 + 12x^2 - 2x - 2 = 12(x+1)(x - 1/\sqrt{6})(x + 1/\sqrt{6})$$





$$f' = 12x^3 + 12x^2 - 2x - 2$$

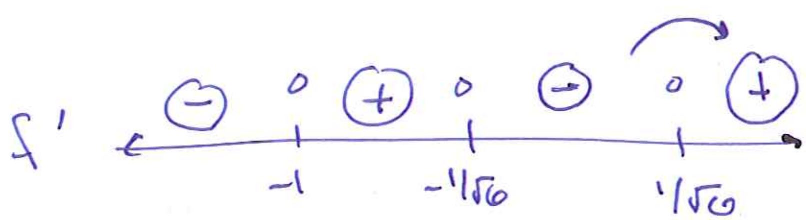
$$= 12x^2(x+1) - 2(x+1)$$

$$= (x+1)(12x^2 - 2)$$

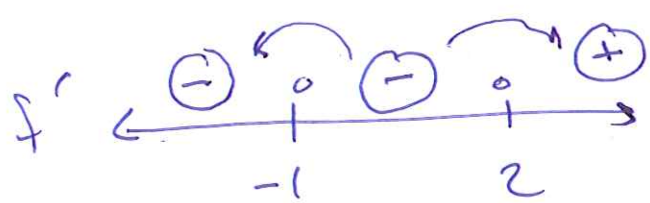
$$= 12(x+1)(x^2 - 1/6) = 12(x+1)\left(x+\tfrac{1}{\sqrt{6}}\right)\left(x-\tfrac{1}{\sqrt{6}}\right)$$

$$2 < \sqrt{6} < 3$$

$$\frac{1}{3} < \frac{1}{\sqrt{6}} < \frac{1}{2}$$



ex $f' = (x+1)^2(x-2)$



Example

Find the local extrema of $f(x) = \dfrac{x^4 + 1}{x^2}$ over the whole real line.

[Hint: Make sure to write the derivative like $f'(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.]

$$f = \frac{x^4+1}{x^2}$$

(quotient rule)

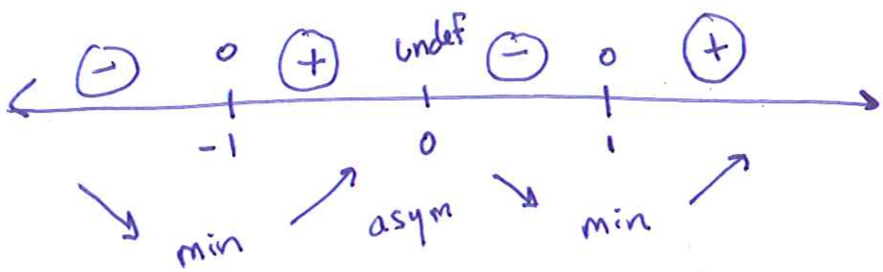
so $$f' = \frac{(4x^3)(x^2) - (x^4+1)(2x)}{x^4} = \frac{4x^5 - 2x^5 - 2x}{x^4}$$

-or- (product rule)

$$= 4x^3 \cdot \frac{1}{x^2} + (x^4+1)\left(-\frac{2}{x^3}\right)$$

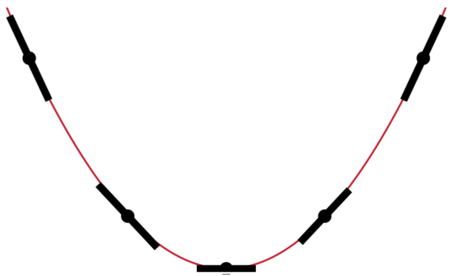
$$= \frac{4x^4 - 2x^4 - 2}{x^3} = 2\,\frac{x^4-1}{x^3}$$

$$= 2\,\frac{(x+1)(x-1)(x^2+1)}{x^3}$$

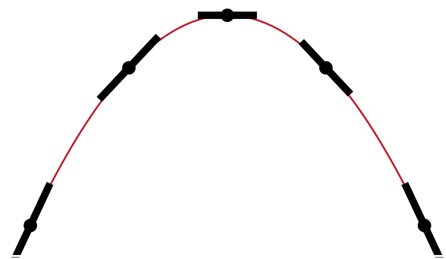


Concavity

Q. How can we measure when a function is concave up or down?



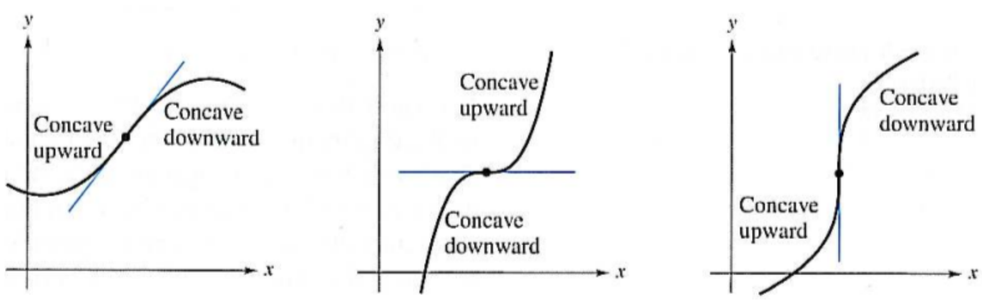
Concave up
$f'(x)$ is increasing
$\boxed{f''(x) > 0}$



Concave down
$f'(x)$ is decreasing
$\boxed{f''(x) < 0}$

Concavity and Inflection Points

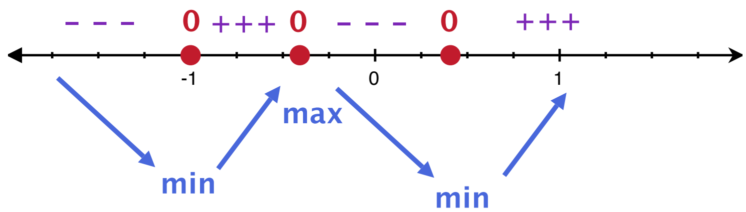
Definition: The function f has an **inflection point** at the point $x = c$ if $f(c)$ exists and the concavity changes at $x = c$ from up to down or vice versa.



Back to the example $f(x) = 3x^4 + 4x^3 - x^2 - 2x$

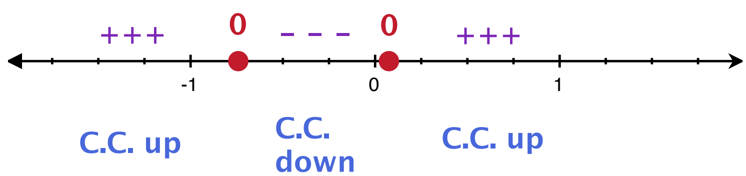
Find the inflection points of $f(x)$, and where $f(x)$ is concave up or down.

We calculated $f'(x) = 12x^3 + 12x^2 - 2x - 2$.

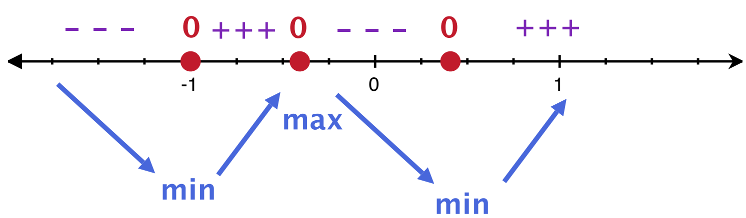


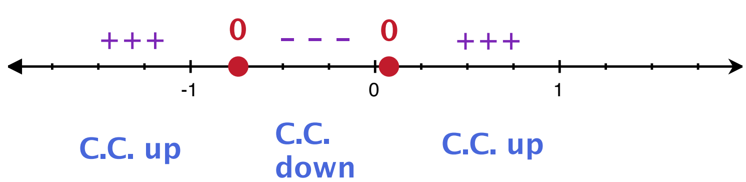
So

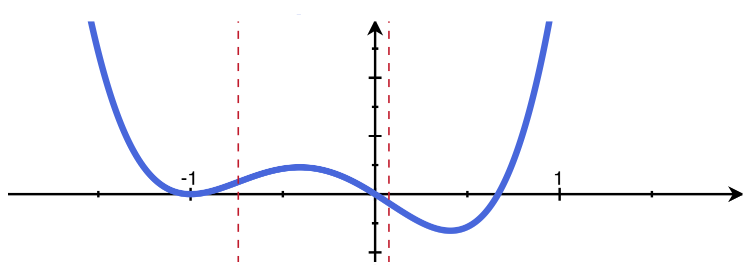
$$f''(x) = 36x^2 + 24x - 2 = (6x - (\sqrt{6} + 2))(6x + \sqrt{6} + 2)$$

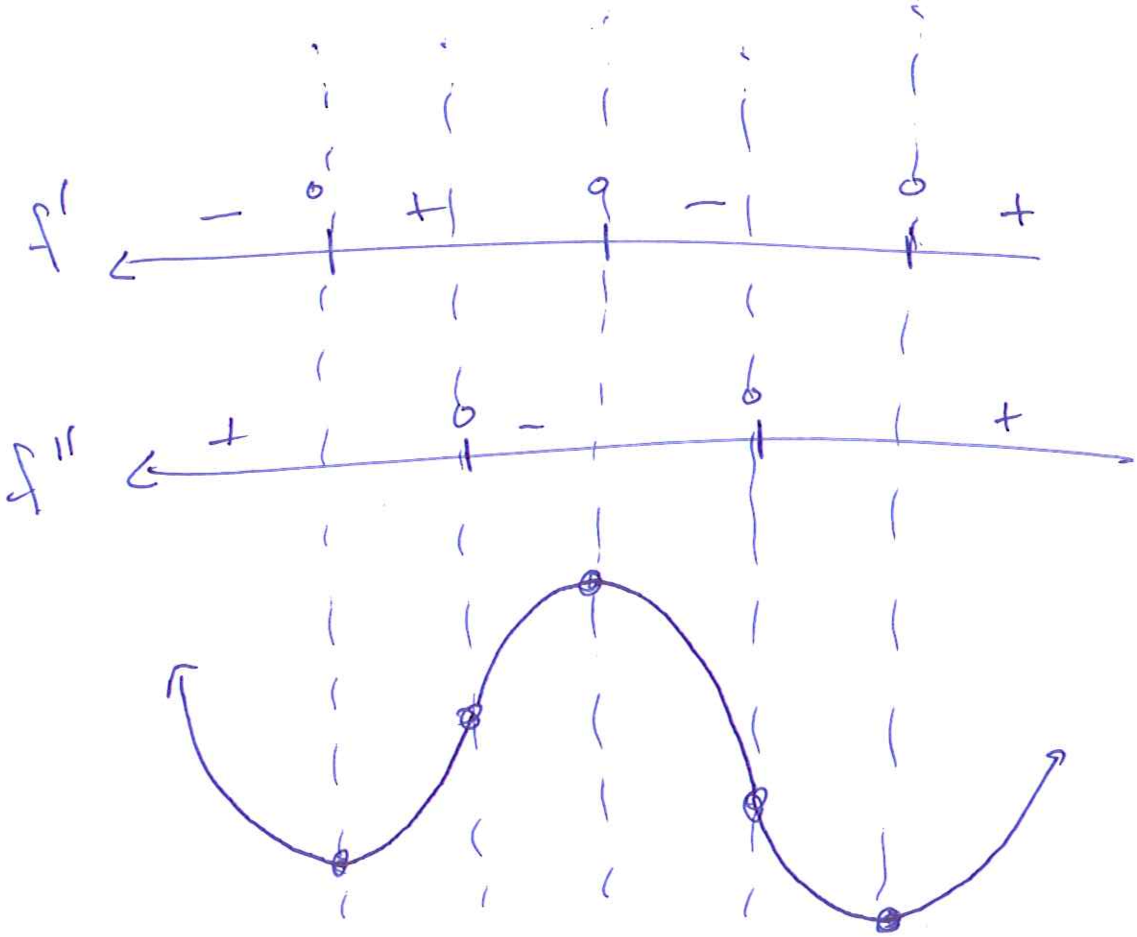


Putting it together

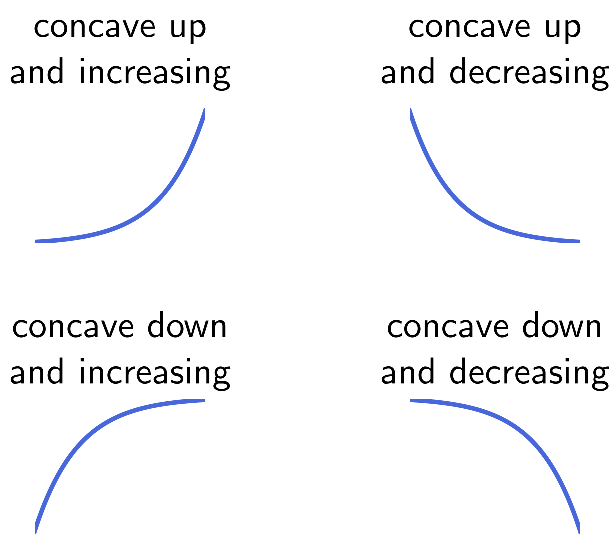







f'
−
0
+
0
−
0
+
f''
+
0
−
0
+

What the pieces look like

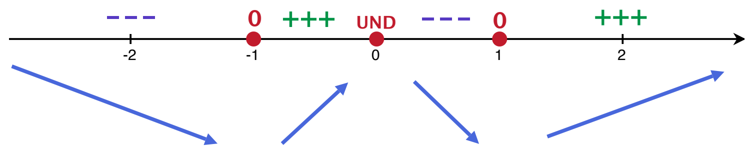


Last elements of graphing

Back to the example where $f(x) = \frac{x^4+1}{x^2}$:

Step 0: Domain. $f(x)$ is defined everywhere except $x = 0$
Step 1: Increasing/decreasing.

We found $f'(x) = \frac{2(x^4-1)}{x^3} = 2\frac{(x^2+1)(x+1)(x-1)}{x^3}$



Step 2: Concavity.

$$f''(x) = 2\frac{(4x^3)(x^3) - (x^4-1)(3x^2)}{x^6} = 2\frac{4x^6 - 3x^6 + 3x^2}{x^6} = 2\frac{x^4+3}{x^4}$$

Therefore $f''(x)$ is always positive, but is undef. at 0.
So $f(x)$ is always **concave up**

Last elements of graphing

$f(x) = \frac{x^4+1}{x^2}$ continued...

Step 4: Extreme behavior.

(a) What is $\lim_{x\to-\infty} f(x)$? What is $\lim_{x\to\infty} f(x)$?
Are there any *horizontal* asymptotes?

For example, $\lim_{x\to-\infty} \frac{x^4+1}{x^2} = \infty \qquad \lim_{x\to\infty} \frac{x^4+1}{x^2} = \infty$,

so there are no horizontal asymptotes.

(b) For any hole in the domain $x = a$, what is $lim_{x\to a^-} f(x)$? $lim_{x\to a^+} f(x)$? Are there any *vertical* asymptotes?

For example, $\lim_{x\to 0^-} \frac{x^4+1}{x^2} = \infty \qquad \lim_{x\to 0^+} \frac{x^4+1}{x^2} = \infty$,

so there is a two-sided vertical asymptote.

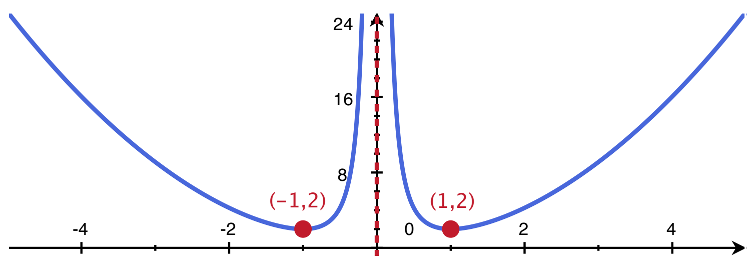
Last elements of graphing

$f(x) = \frac{x^4+1}{x^2}$ continued...

Step 5: Plot salient points.

(a) Find any roots of $f(x)$. (x-intercepts)

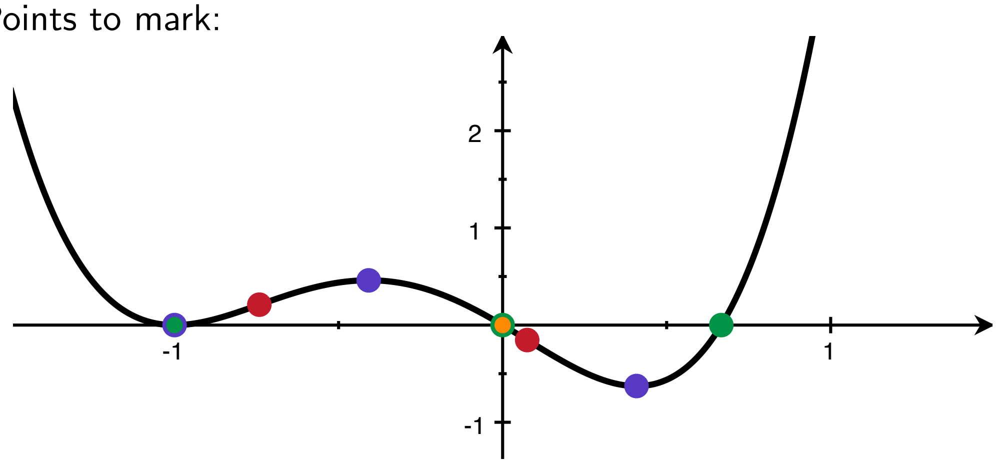
(b) Calculate $f(x)$ at $x = 0$ (y-intercept), critical points, and inflection points.



($f(x)$ doesn't have any roots, and doesn't have any inflection points)

Back to $3x^4 + 4x^3 - x^2 - 2x$

There are no vertical or horizontal asymptotes, and there are no points missing from the domain.
Points to mark:



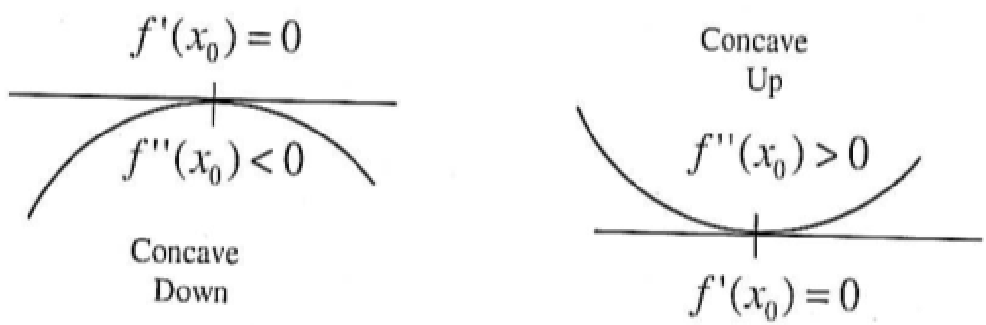
orange = y-intercept
green = roots
purple = critical points
red = inflection points.

The second derivative test

Theorem

Let f be a function whose second derivative exists on an interval I containing x_0.

1. *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f(x_0)$ is a local minimum.*
2. *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f(x_0)$ is a local maximum.*

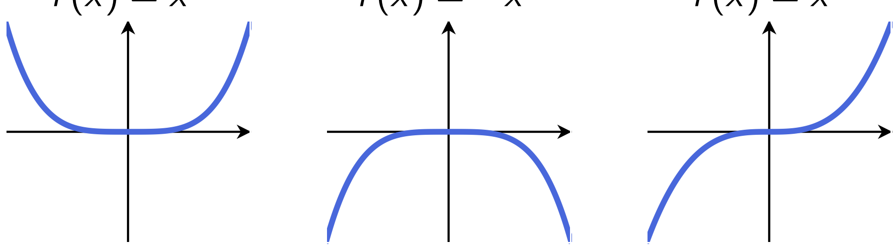


Warning: If $f'(x_0) = 0$ and $f''(x_0) <= 0$, then the test **fails**, use the first derivative test to decide.

Why the 2nd derivative test fails when $f''(x_0) = 0$

If $f'(x_0) = 0$ and $f''(x_0) = 0$, anything can happen!

$f(x) = x^4$	$f(x) = -x^4$	$f(x) = x^3$



$f'(x) = 4x^3$	$f'(x) = -4x^3$	$f'(x) = 3x^2$
$f''(x) = 12x^2$	$f''(x) = -12x^2$	$f''(x) = 6x$
	so	
$f'(0) = 0$	$f'(0) = 0$	$f'(0) = 0$
$f''(0) = 0$	$f''(0) = 0$	$f''(0) = 0$

(The second derivative being zero just means the function is almost flat.)

Sketch graphs of the following functions:

1. $f(x) = -3x^5 + 5x^3$.
2. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

Instructions:

* Find any places where $f(x)$ is 0 or undefined.
* Calculate $f'(x)$ and find critical/singular points.
* Classify where $f'(x)$ is positive/negative, and therefore where $f(x)$ is increasing/decreasing.
* Calculate $f''(x)$, and find where it's 0 or undefined.
* Classify where $f''(x)$ is positive/negative, and therefore where $f(x)$ is concave up/down.
* Calculate $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ for anything where $f(a)$ is undefined.
* Calculate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ to see what the tails are doing.

Hint for 2: Always simplify as a fraction of polynomials after taking a derivative.

①

$$f(x) = -3x^5 + 5x^3 = x^3(-3x^2+5) = -3(x^3)\left(x^2 - \tfrac{5}{3}\right)$$

$f = 0$ @ $x = 0$, $x = \pm\sqrt{5/3}$

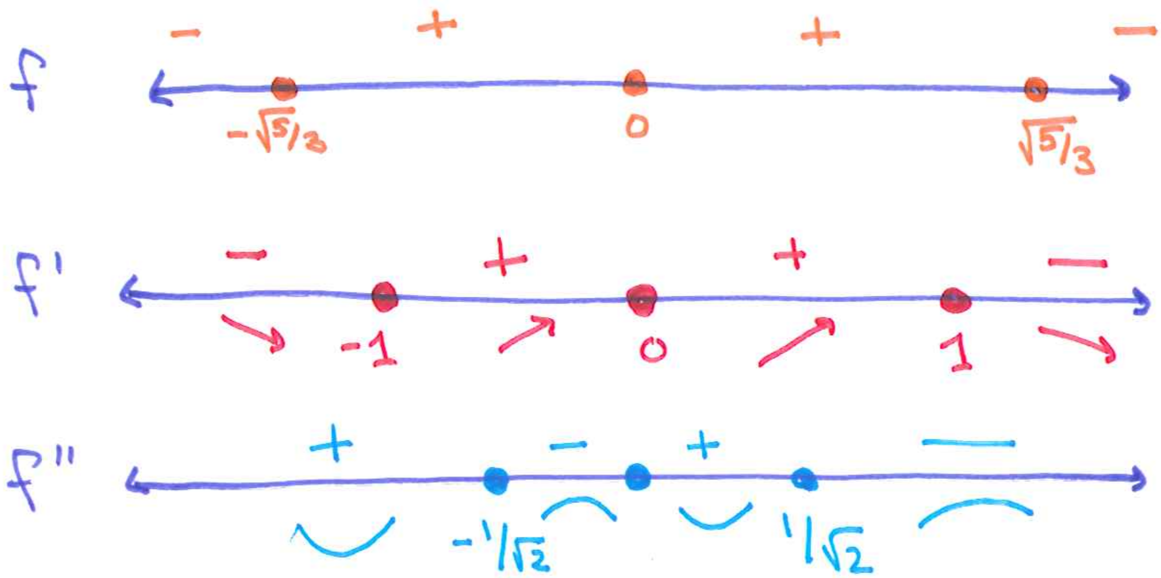
$$f'(x) = -15x^4 + 15x^2 = -15(x^4 - x^2)$$

$$= -15x^2(x+1)(x-1)$$

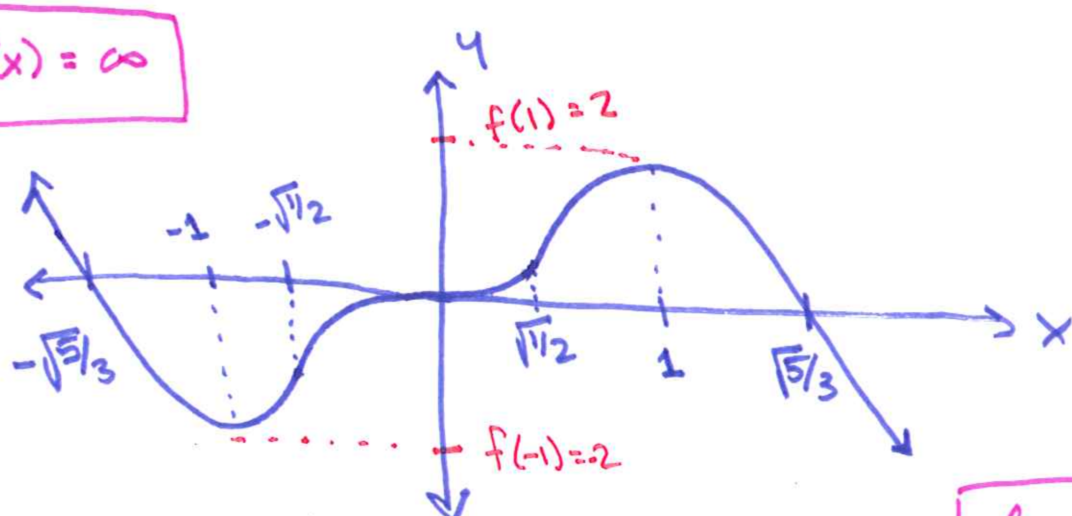
$f' = 0$ @ $x = 0, \pm 1$

$$f''(x) = -60x^3 + 30x = -60(x)\left(x^2 - \tfrac{1}{2}\right)$$

$f'' = 0$ @ $x = 0$, $x = \pm\sqrt{1/2}$



$$\lim_{x\to-\infty} f(x) = \infty$$



$$\lim_{x\to\infty} f(x) = -\infty$$

② $f(x) = \dfrac{x^2-1}{x^2+1} = \dfrac{(x+1)(x-1)}{x^2+1}$

so $f(x) = 0$ when $\boxed{x = \pm 1}$
(and is defined everywhere
since $x^2+1 > 0$)

$$f'(x) = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}$$

$f'(x) = 0$ when $\boxed{x=0}$
(and is defined everywhere)

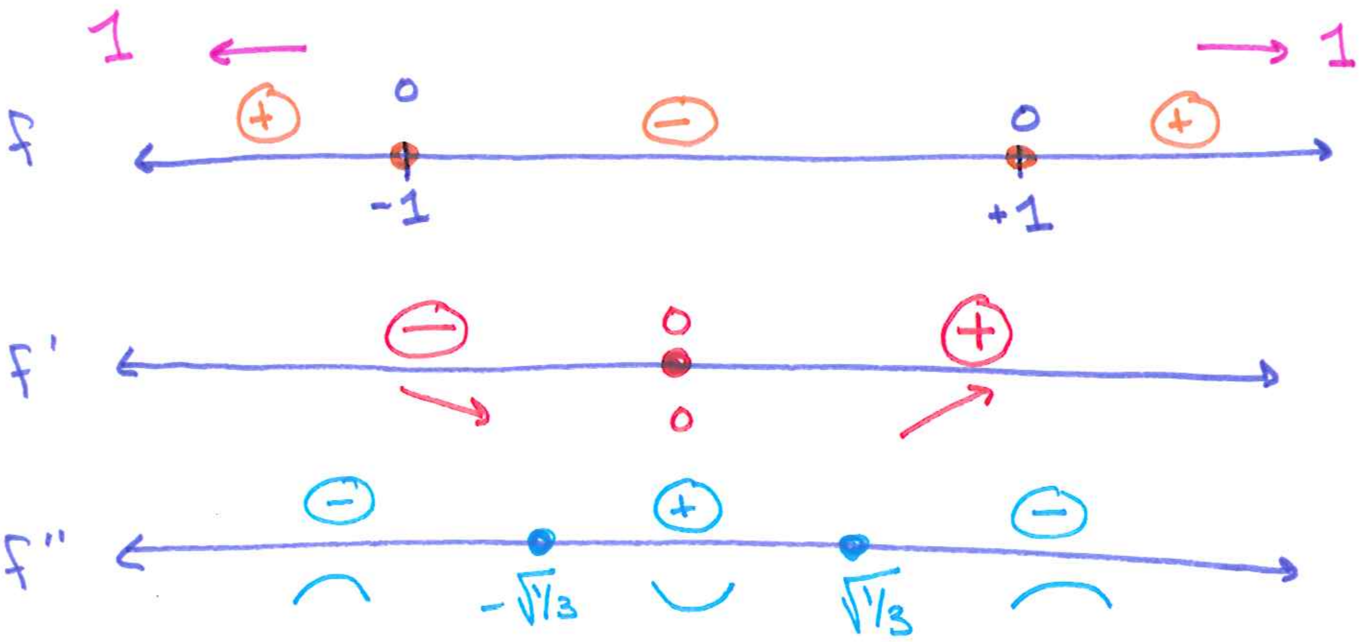
$$f''(x) = \frac{4(x^2+1)^2 - 4x \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4}$$

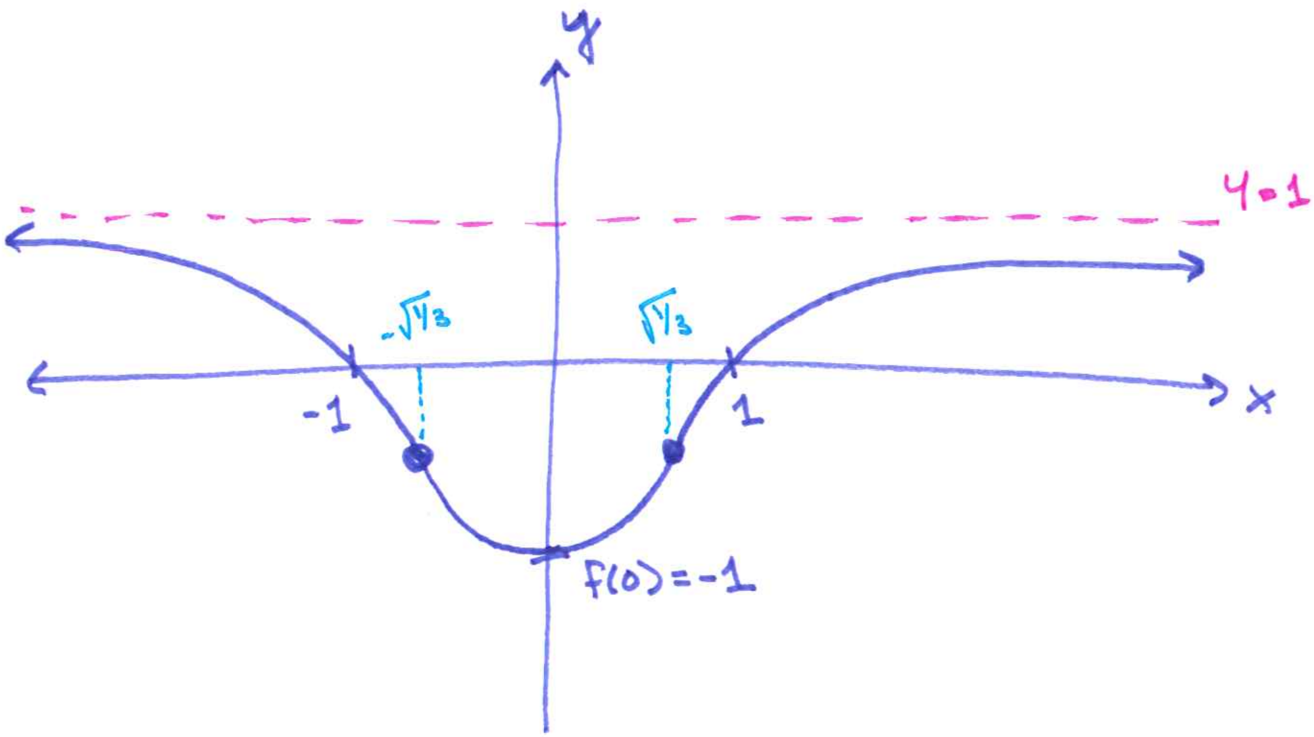
$$= \frac{(x^2+1)(4)(x^2+1-4x^2)}{(x^2+1)^4}$$

$$= \frac{-4 \cdot 3\,(x^2 - 1/3)}{(x^2+1)^4} = -12\,\frac{(x+1/\sqrt{3})(x-1/\sqrt{3})}{(x^2+1)^4}$$

so $f''(x) = 0$ when $\boxed{x = \pm 1/\sqrt{3}}$

$\lim_{x \to \infty} \dfrac{x^2-1}{x^2+1} = 1$ $\lim_{x \to -\infty} \dfrac{x^2-1}{x^2+1} = 1$


1
1
f
0
0
-1
+1
f'
0
0
f''
$-\sqrt{1/3}$
$\sqrt{1/3}$


y
y=1
$-\sqrt{1/3}$
$\sqrt{1/3}$
-1
1
x
f(0)=-1

Extra practice: Graphing functions

For each of the following graphing problems also determine (a) where $f(x)$ is defined, (b) where $f(x)$ is continuous, (c) where $f(x)$ is differentiable, (d) where $f(x)$ has vertical asymptotes, (e) if $f(x)$ has horizontal asymptotes, (f) where $f(x)$ is increasing and where it is decreasing, (g) where $f(x)$ is concave up and where it is concave down, (h) what the critical points of $f(x)$ are, and (i) where the points of inflection are. Then sketch a graph.

1. Graph $f(x) = a$, where a is a constant.
2. Graph $f(x) = ax + b$, where a and b are constants.
3. Graph $f(x) = a(x - c) + b$, where a, b and c are constants.
4. Graph $f(x) = \begin{cases} 2 - x, & \text{if } x \geq 1, \\ x, & \text{if } 0 \leq x \leq 1. \end{cases}$
5. Graph $f(x) = \begin{cases} 2 + x, & \text{if } x \geq 0, \\ 2 - x, & \text{if } x < 0. \end{cases}$
6. Graph $f(x) = \begin{cases} 1 - x, & \text{if } x < 1, \\ x^2 - 1, & \text{if } x \geq 1. \end{cases}$
7. Graph $f(x) = 2x - x^2$.
8. Graph $f(x) = x - x^2 - 27$.
9. Graph $f(x) = 3x^2 - 2x - 1$.
10. Graph $f(x) = x^3 - x + 1$.
11. Graph $f(x) = x^3 - x - 1$.
12. Graph $f(x) = (x - 2)^2(x - 1)$.
13. Graph $f(x) = 2x^3 - 21x^2 + 36x - 20$.
14. Graph $f(x) = 2x^3 + x^2 - 20x$.
15. Graph $f(x) = 1 - x^4$.
16. Graph $f(x) = x^4 - 3x^2 + x$.
17. Graph $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.
18. Graph $f(x) = 3x^4 - 16x^3 + 18x^2$.
19. Graph $f(x) = 3x^5 - 25x^3 + 60x$.
20. Graph $f(x) = x^5 - 4x^4 + 4x^3$.
21. Graph $f(x) = x^3(x - 2)^2$.
22. Graph $f(x) = (x - 2)^4(x + 1)^3(x - 1)$.
23. Graph $f(x) = (x - 3)^5(x + 1)^4$.
24. Graph the function $f(x)$ such that $\dfrac{df}{dx} = 1/x$ and $f(-1) = 2$ and $f(1) = 1$.
25. Graph $f(x) = x + 1/x$.
26. Graph $f(x) = \dfrac{x^2 + 2x - 20}{x - 4}$.
27. Graph $f(x) = \dfrac{1}{x^2 + 1}$.
28. Graph $f(x) = \dfrac{x^3}{x^2 + 1}$.
29. Graph $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$.
30. Graph $f(x) = \dfrac{2x^2}{x^2 - 1}$.
31. Graph $f(x) = \dfrac{x^2 + 7x + 3}{x^2}$.
32. Graph $f(x) = \dfrac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4}$.
33. Graph $f(x) = \dfrac{x^2 - 1}{x^3 - 4x}$.