Going between graphs of functions and their derivatives:

Mean value theorem, Rolle's theorem, and intervals of increase and decrease

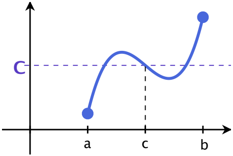
Recall: The *Intermediate* Value Theorem

Suppose f is continuous on a closed interval $[a, b]$.

If $\quad f(a) < C < f(b) \quad$ or $\quad f(a) > C > f(b)$,

then there is at least one point c in the interval $[a, b]$ such that

$$f(c) = C.$$

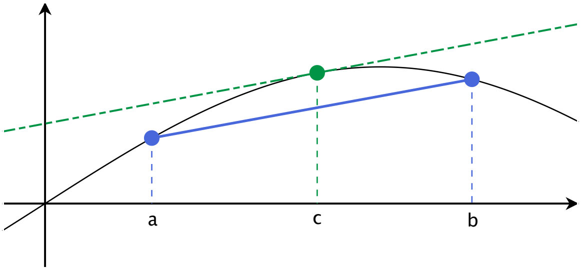


The *Mean* Value Theorem

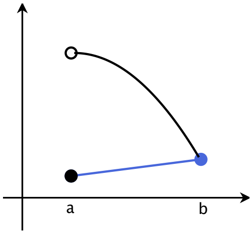
Theorem

Suppose that f is defined and continuous on a closed interval $[a, b]$, and suppose that f' exists on the open interval (a, b). Then there exists a point c in (a, b) such that

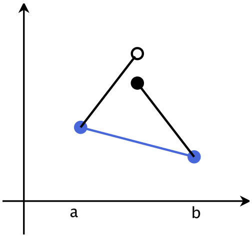
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



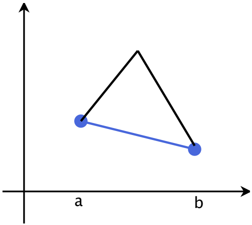
Bad examples



Discontinuity
at an endpoint



Discontinuity
at an interior point



No derivative
at an interior point

Examples

Does the mean value theorem apply to $f(x) = |x|$ on $[-1, 1]$?

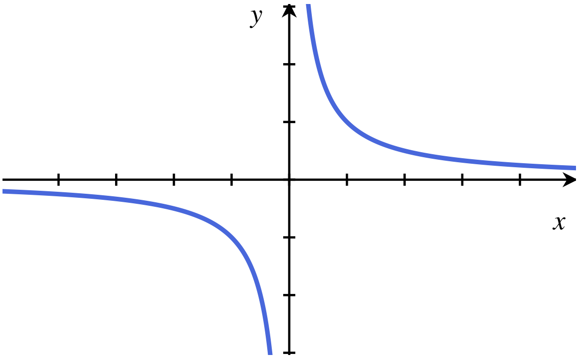
(No! Because $f(x)$ is not differentiable at $x = 0$.)

How about to $f(x) = |x|$ on $[1, 5]$?

(Yes! Because $f(x) = x$ on this domain, which is differentiable.)

Example

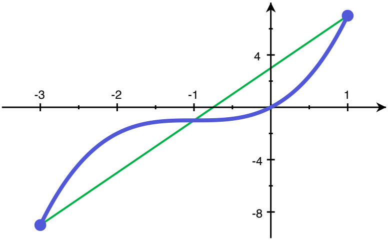
Under what circumstances does the Mean Value Theorem apply to the function $f(x) = 1/x$?



ANY closed interval on the domain!

Example

Verify the conclusion of the Mean Value Theorem for the function $f(x) = (x+1)^3 - 1$ on the interval $[-3, 1]$.



Step 1: Check that the conditions of the MVT are met.

Step 2: Calculate the slope m of the line joining the two endpoints.

Step 3: Solve the equation $f'(x) = m$.

1. polynomials are cont. & diff'ble <u>everywhere</u>. ✓

2. $$\frac{f(1) - f(-3)}{1-(-3)} = \frac{[(1+1)^3 - 1] - [(-3+1)^3 - 1]}{4}$$

$$= \frac{(8-1) - (-8-1)}{4}$$

$$= 16/4 = \boxed{4 = m}$$

3. Solve

$$4 = f'(x) = 3(x+1)^2$$

$$x+1 = \pm\sqrt{4/3}$$

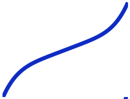
$$x = \boxed{\pm\frac{2}{\sqrt{3}} - 1}$$

↓ btwn 1 & 2

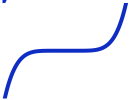
Intervals on increase/decrease

Formally,

f is *increasing* if
$f(x_1) < f(x_2)$ whenever $x_1 < x_2$.



f is *nondecreasing* if
$f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$.



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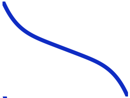
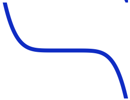
f is *decreasing* if
$f(x_1) > f(x_2)$ whenever $x_1 < x_2$.



f is *nonincreasing* if
$f(x_1) \geq f(x_2)$ whenever $x1 < x2$.



Sign of the derivative

If $f(x)$ is **increasing**, what is the sign of the derivative?
Look at the difference quotient:

$$\frac{f(x+h)-f(x)}{h}$$

The derivative is a two-sided limit, so we have two cases:

Case 1: h is positive.
So $x+h>x$, which implies $f(x+h)-f(x)>0$.
So

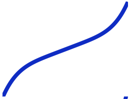
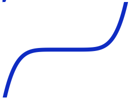
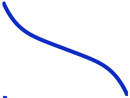
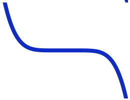
$$\frac{f(x+h)-f(x)}{h}>0.$$

Case 2: h is negative.
So $x+h<x$, which implies $f(x+h)-f(x)<0$.
So

$$\frac{f(x+h)-f(x)}{h}>0.$$

So the difference quotient is positive!

Intervals on increase/decrease

		$\frac{f(x+h)-f(x)}{h}$	$\lim_{h\to 0} \sim$
Formally,			
f is *increasing* if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.		pos.	pos. or 0 (non-neg)
f is *nondecreasing* if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$.		non-neg.	non-neg.
f is *decreasing* if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.		neg.	non-pos.
f is *nonincreasing* if $f(x_1) \geq f(x_2)$ whenever $x1 < x2$.		non-pos.	non-pos.

So we can calculate some of the "shape" of $f(x)$ by knowing when its derivative is positive, negative, and 0!

Example

On what interval(s) is the function $f(x) = x^3 + x + 1$ increasing or decreasing?

Step 1: Calculate the derivative.

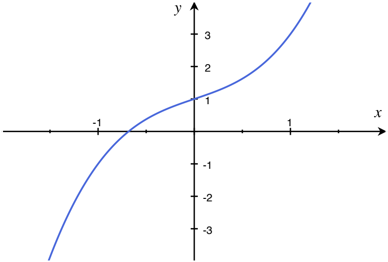
$$f'(x) = 3x^2 + 1$$

Step 2: Decide when the derivative is positive, negative, or zero.

$f'(x)$ is always positive!

Step 3: Bring that information back to $f(x)$.

$f(x)$ is always increasing!



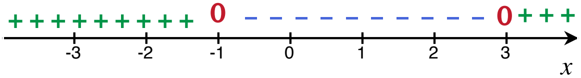
Example

Find the intervals on which the function $f(x) = 2x^3 - 6x^2 - 18x + 1$ is increasing and those on which it is decreasing.

Step 1: Calculate the derivative.

$$f'(x) = 6x^2 - 12x - 18 = 6(x-3)(x+1)$$

Step 2: Decide when the derivative is positive, negative, or zero.



Step 3: Bring that information back to $f(x)$.

$f(x)$ is increasing, then decreasing, then increasing.

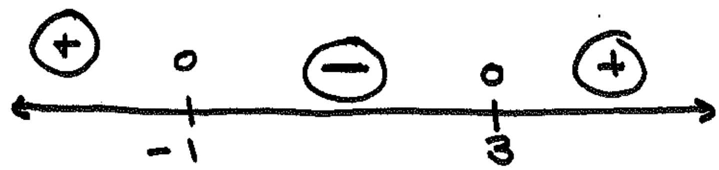
Example

Find the intervals on which the function $f(x) = 2x^3 - 6x^2 - 18x + 1$ is increasing and those on which it is decreasing.

Step 1: Calculate the derivative.

$$f' = 6x^2 - 12x - 18 = 6(x^2 - 2x - 3) = 6(x-3)(x+1)$$

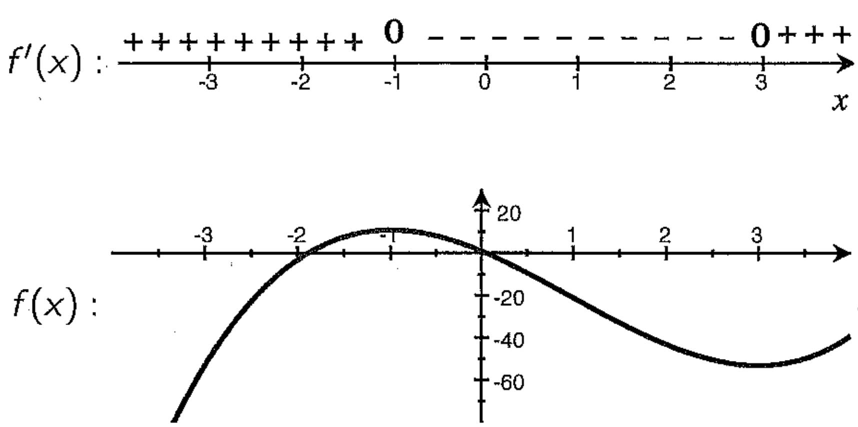
Step 2: Decide when the derivative is positive, negative, or zero.



Step 3: Bring that information back to $f(x)$.

incr $(-\infty, -1] \cup [3, \infty)$

decr $[-1, 3]$

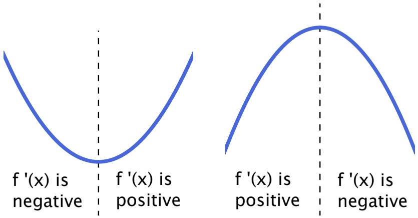


If f is continuous on a closed interval $[a, b]$, then there is a point in the interval where f is largest (**maximized**) and a point where f is smallest (**minimized**).

The maxima or minima will happen either

1. at an endpoint, or
2. at a **critical point**, a point c where $f'(c) = 0$

what's going on right before c?
what's going on right after c?

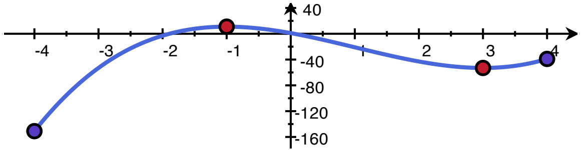


Example

For the function $f(x) = 2x^3 - 6x^2 - 18x + 1$, let us find the points in the interval $[-4, 4]$ where the function assumes its maximum and minimum values.

$$f'(x) = 6x^2 - 12x - 18 = 6(x-3)(x+1)$$

x	$f(x)$
-1	11
3	-53
-4	-151
4	-39



Rolle's Theorem

Theorem

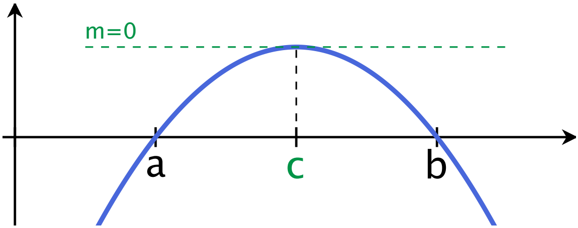
Suppose that the function f is

continuous *on the closed interval* $[a, b]$,

differentiable *on the open interval* (a, b), *and*

a *and* b *are both* **roots** *of* f.

Then there is at least one point c *in* (a, b) *where* $f'(c) = 0$.



(In other words, if g didn't jump, then it had to turn around)

Back to Newton's method

Remember: Newton's method helped us fine roots of functions.

> Pick an x_0 to start. To get x_{i+1}, follow the tangent line to $f(x)$ at x_i down to it's x-intercept. The x_i's get closer and closer to a root of f.

But how do we know when we've found all of them?

For example: Find the roots of $f(x) = x^5 - 3x + 1$.

If x_0 is...	-2	-1	0	1	2	
then the x_i's get closer to...						
	-1.3888	-1.3888	0.3347	1.2146	1.2146	

$x_0 =$	-.9	-.8	-.7	-.6	.5	.6	.7
$x_i \to$	-1.3...	1.2 ...	1.2...	0.3...	0.3...	0.3...	0.3...
$x_0 =$	-10	-20	-50	-100	-1000	-10000	
$x_i \to$	-1.3...	-1.3...	-1.3...	-1.3...	-1.3...	-1.3...	
$x_0 =$	10	20	50	100	1000	10000	100000
$x_i \to$	1.2...	1.2...	1.2...	1.2...	1.2...	1.2...	1.2...

After plugging in lots of x_0's, we've only found three roots. But there could be up to 5! How do we know we're not just very unlucky?

Use Rolle's Theorem to show that $f(x) = x^5 - 3x + 1$ has exactly three real roots!

Step 1: Show that there are at *most* three roots.

Step 2: Show that there are at *least* three roots.

Two methods:

(1) Use Newton's method to root out three roots, or

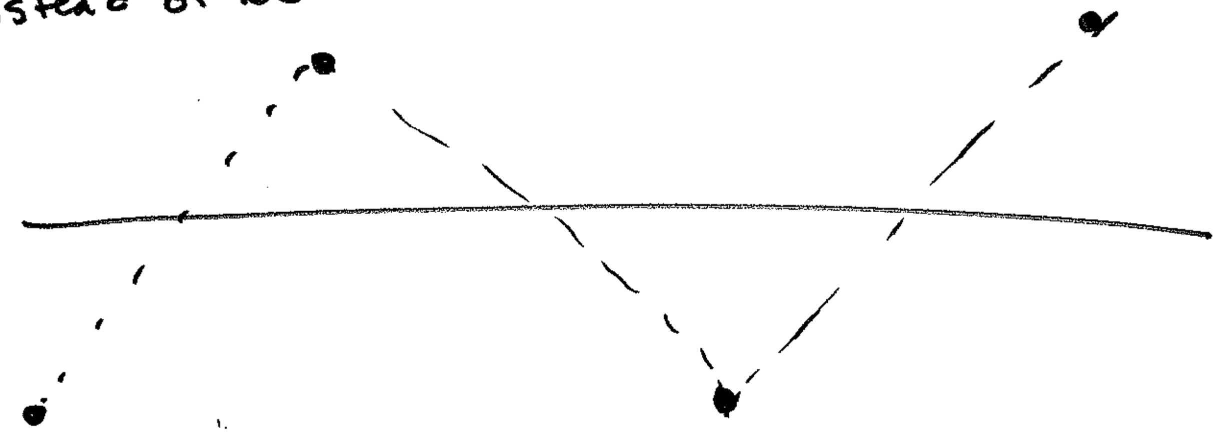
(2) find four points $f(x)$ which alternate signs, and use the intermediate value theorem.

(IVT: If $f(x)$ is cont. and $f(a) < C < f(b)$, then there's a c btwn. a and b where $f(c) = C$)

On your own:

1. **Do an analysis of increasing/decreasing on $f(x)$.**
   How many times does $f(x)$ turn around?
   Conclude: what is an upper bound on the number of roots?
2. **Find the heights of the critical points.**
   Using the intermediate value theorem, what is a lower bound on the number of roots? Can you do better if you also find the height of the function at a big positive number and a big negative number?
3. **Conclude: How many real roots does $f(x)$ have?**
4. **Bonus:**
   Using the approximations from before, sketch a graph.

instead of Newton's: find these pts



$f' = 5x^4 - 3 = 0$

$x^4 = \frac{3}{5}$

$$\boxed{x = \pm\left(\frac{3}{5}\right)^{1/4}}$$

two real solutions.

$\Rightarrow$ at most 3 roots!