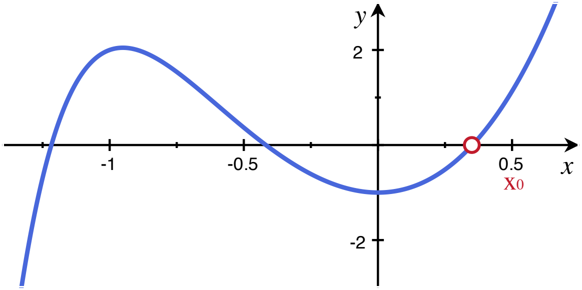
Curves are tricky. Lines aren't.

~~Newton's Method and Linear Approximations~~

Newton's Method for finding roots

Goal: Where is $f(x) = 0$?

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$



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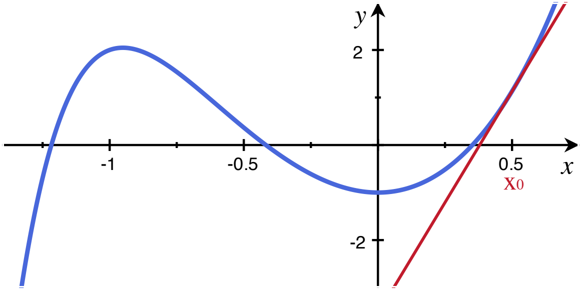

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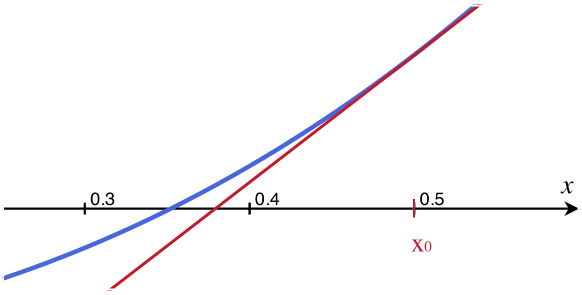

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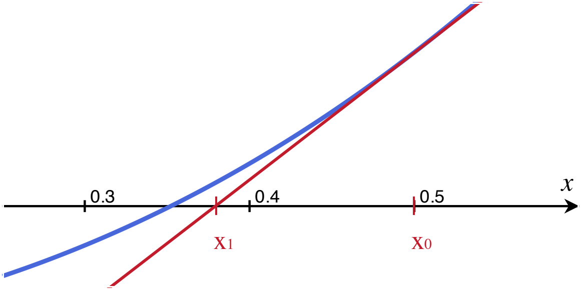

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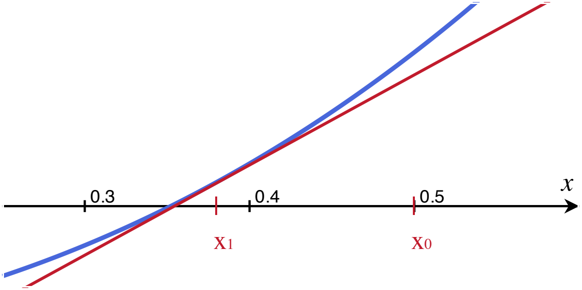

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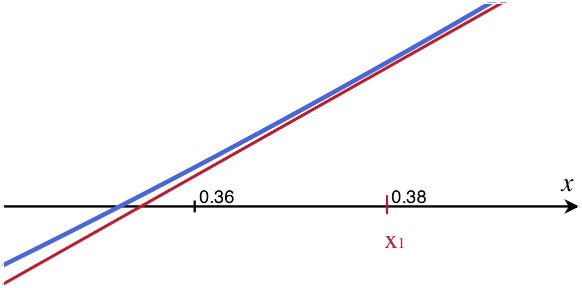

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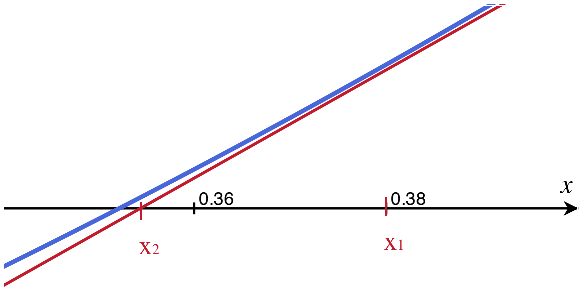


$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$

$$f'(x) = 7x^6 + 9x^2 + 14x$$

i	x_i	$f(x_i)$	$f'(x_i)$	tangent line	x-intercept
0	0.5	1.133	9.359	$y = 1.133 + 9.359(x - 0.5)$	0.379
1	0.379	0.170	6.619	$y = 0.170 + 6.619(x - 0.379)$	0.353
2	0.353	0.007	6.084	$y = 0.007 + 6.084(x - 0.353)$	0.352
3	0.352	0.00001	6.060	$y = 0.00001 + 6.060(x - 0.352)$	0.352

Newton's Method

Step 1: Pick a place to start. Call it x_0.

Step 2: The tangent line at x_0 is $y = f(x_0) + f'(x_0) * (x - x_0)$. To find where this intersects the x-axis, solve

$$0 = f(x_0) + f'(x_0) * (x - x_0) \quad \text{to get} \quad x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This value is your x_1.

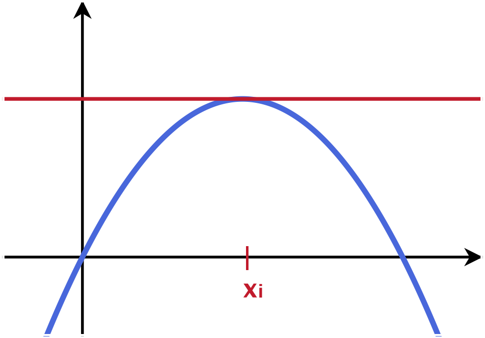
Step 3: Repeat with your new x-value. In general, the 'next' value is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Step 4: Keep going until your x_i's stabilize.
What they stabilize to is an approximation of your root!

Caution!

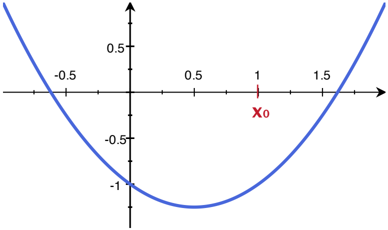
Bad places to pick: Critical points! (where f'(x)=0)



Tangent line has no x-intercept!

Even *near* critical points, the algorithm goes much slower.
Just stay away!

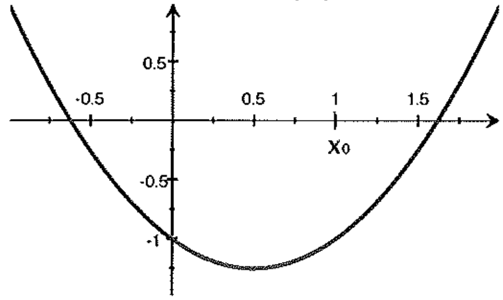
You try: Approximate a root of $f(x) = x^2 - x - 1$ near $x_0 = 1$.



$$f'(x) = 2x - 1$$

i	x_i	$f(x_i)$	$f'(x_i)$	$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
0	1	-1	1	2
1	2	1	3	$5/3 \approx 1.667$
2	$5/3$	$1/9$	$7/3$	$34/21 \approx 1.619$

You try: Approximate a root of $f(x) = x^2 - x - 1$ near $x_0 = 1$.



$f'(x) = 2x - 1$

i	x_i	$f(x_i)$	$f'(x_i)$	$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
0	1			
1				
2				

$f(1) = 1 - 1 - 1 = \boxed{-1}$ $\qquad f'(1) = 2 - 1 = \boxed{1}$

$x_1 = 1 - {-1}/{1} = 1 + 1 = \boxed{2}$

$f(2) = 4 - 2 - 1 = \boxed{1}$ $\qquad f'(2) = 4 - 1 = 3$

$x_2 = 2 - 1/3 = \frac{6-1}{3} = \boxed{5/3}$

$f(5/3) = \frac{25}{9} - \frac{5}{3} - 1 = \frac{25 - 15 - 9}{9} = \boxed{1/9}$

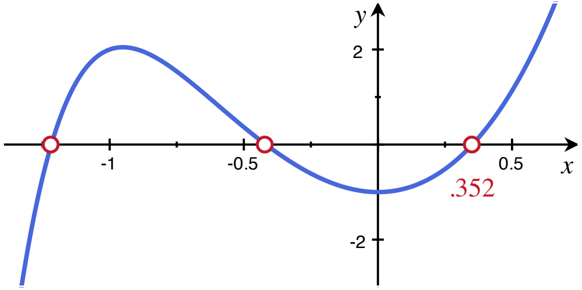
$f'(5/3) = \frac{10}{3} - 1 = \frac{10-3}{3} = \boxed{7/3}$

$x_3 = 5/3 - {}^{1/9}/_{7/3} = 5/3 - \frac{3}{7 \cdot 9} = \boxed{\frac{34}{7 \cdot 3}}$

Back to the example:

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$

$$f'(x) = 7x^6 + 9x^2 + 14x$$



$r_1 \approx$ $r_2 \approx$ $r_3 \approx 0.352$

Newton's Method

Select or Enter a function

(x^2-25)*(x^2-4)*x/60

x^7+3*x^3+7*x^2-1

Instructions

Newton's method searches for a root of f(x) = 0, given an initial guess x0. The success of the search depends on a suitable starting guess. Newton's method generates a sequence x0, x1, x2, ..., of approximations that generally converge rapidly to a root.

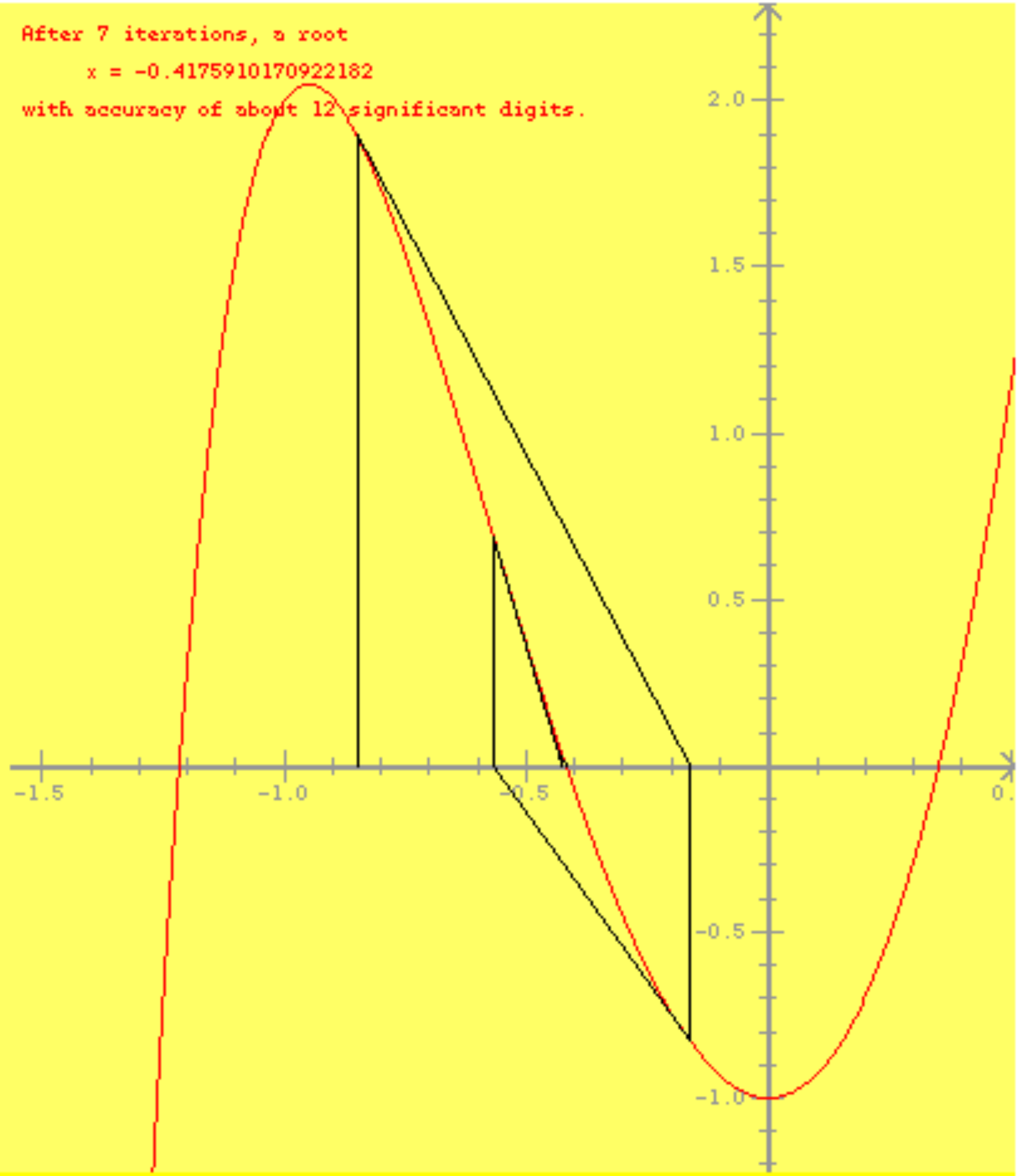
It might be expected that Newton's Method will behave badly at a root r if f '(r) = 0 (note that f ' appears in the denominator in the following formula). It does, which is why we used the term 'generally' above. Convergence is often very slow at roots where the derivative is zero.

The key step in the iteration is the formula:

$$x(n+1) = x(n) - f(x(n))/f'(x(n)),$$

where f '(x) is the derivative of f(x). This applet implements Newton's method, beginning with a starting guess that the user selects with a mouse click or by typing it in the text field provided.

x	f(x)
-.850000000	1.894547912
-.162935654	-.827143719
-.567994011	.689514028
-.424742516	.030470971
-.417625854	.000147703
-.417591018	.000000004
-.417591017	.000000000



For starting value, click the mouse or type a value: -.85

Zoom X | Zoom Y | Reset Zoom

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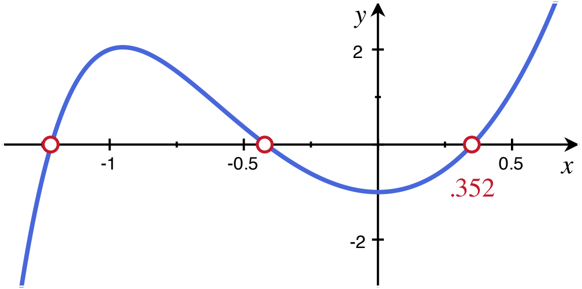
Refer to authors' book. Principles of Calculus Modeling: An Interactive Approach

Back to the example:

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$

$$f'(x) = 7x^6 + 9x^2 + 14x$$

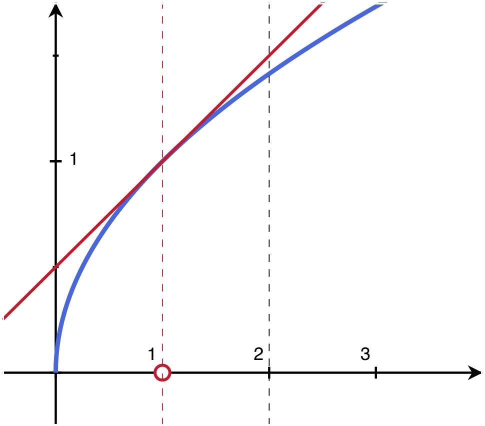


$r_1 \approx -1.217$ $r_2 \approx -0.418$ $r_3 \approx 0.352$

Linear approximations of functions

Goal: approximate functions

Example: approximate $\sqrt{2}$

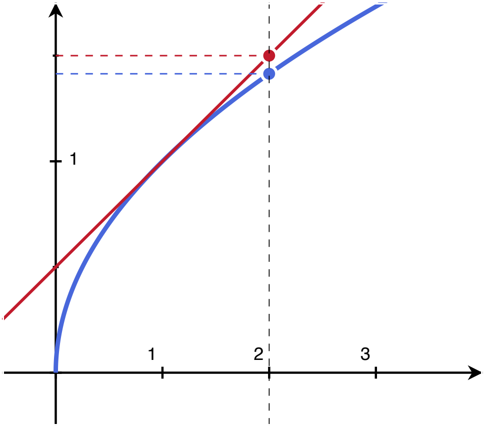


$$y = 1 + \frac{1}{2}(x - 1)$$

Linear approximations of functions

Goal: approximate functions

Example: approximate $\sqrt{2}$



$$y = 1 + \frac{1}{2}(x - 1)$$

$$\sqrt{2} \approx 1 + \frac{1}{2}(2 - 1) = 1.5 \qquad (\sqrt{2} = 1.414...)$$

Linear approximations

If $f(x)$ is differentiable at a, then the tangent line to $f(x)$ at $x = a$ is

$$y = f(a) + f'(a) * (x - a).$$

For values of x *near* a, then

$$f(x) \approx f(a) + f'(a) * (x - a).$$

This is the *linear approximation* of f about $x = a$. We usually call the line $L(x)$.

Approximate $\sqrt{5}$:

Our last approximation told us

$$\sqrt{5} \approx L(5) = 1 + \frac{1}{2}(5 - 1) = 3$$

This isn't great... $(3^2 = 9)$

Better: Use the linear approximation about $x = 4$!

The tangent line is

$$L(x) = 2 + \frac{1}{4}(x - 4)$$

so

$$\sqrt{5} \approx L(5) = 2 + \frac{1}{4}(5 - 4) = \boxed{2.25}$$

Better! $(2.25^2 = 5.0625)$

Even better approximations...

The linear approximation is **the** line which satisfies

$$L(a) = f(a) + f'(a)(a-a) = \boxed{f(a)}$$

and

$$L'(a) = \frac{d}{dx}\left(f(a) + f'(a)(x-a)\right) = \boxed{f'(a)}$$

A **better** approximation might be a quadratic polynomial $p_2(x)$ which **also** satisfies $p_2''(a) = f''(a)$:

$$\boxed{p_2(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2}$$

or a cubic polynomial $p_3(x)$ which also satisfies $p_3^{(3)}(a) = f^{(3)}(a)$:

$$\boxed{p_3(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2 + \tfrac{1}{2*3}f^{(3)}(a)(x-a)^3}$$

and so on...

These approximations are called **Taylor polynomials** (read §2.14)

$$P_2(x) = \sqrt{a} + \frac{1}{2\sqrt{a}}(x-a) + \frac{1}{2}\cdot\left(\frac{-1}{4(\sqrt{a})^3}\right)(x-a)^2$$

$f = \sqrt{x}$ $\quad f' = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$ $\quad f'' = -\frac{1}{4}x^{-3/2}$

$f''' = \frac{3}{8}\cdot x^{-5/2} = \frac{3}{8}\cdot\frac{1}{(\sqrt{x})^5}$ $\quad = -\frac{1}{4}\cdot\frac{1}{(\sqrt{x})^3}$

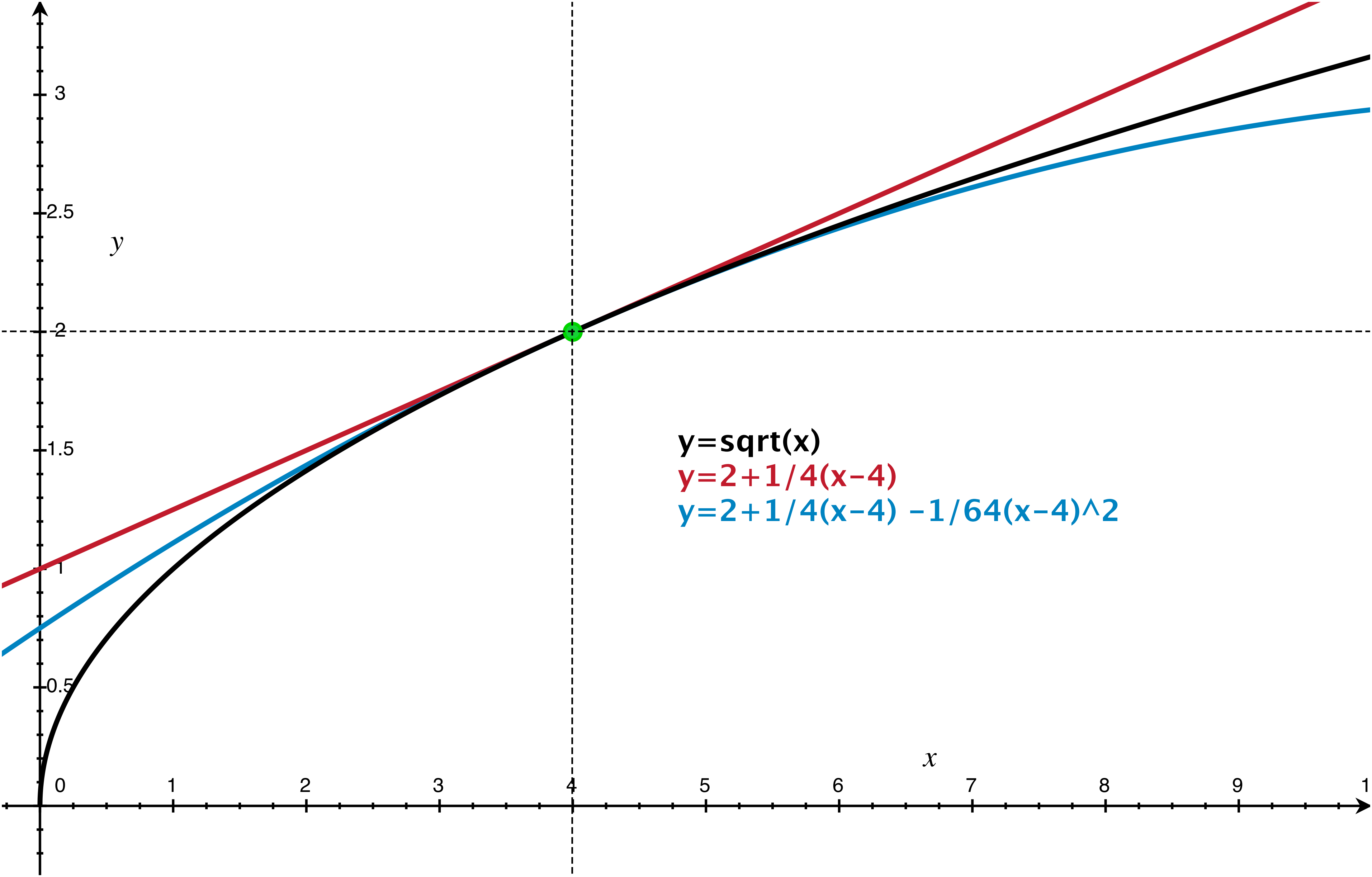
near $a = 4$

$$\sqrt{x} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(x-4) + \frac{1}{2}\cdot\frac{-1}{4(\sqrt{4})^3}(x-4)^2$$

$$= 2 + \frac{1}{4}(x-4) + \frac{-1}{64}(x-4)^2$$

3rd degree term

$$\frac{1}{2\cdot 3}\cdot f'''(4)(x-4)^3 = \frac{1}{16(\sqrt{4})^5}(x-4)^3$$



y=sqrt(x)
y=2+1/4(x−4)
y=2+1/4(x−4) −1/64(x−4)^2
y
x
3
2.5
2
1.5
1
0.5
0 1 2 3 4 5 6 7 8 9 1