

## Using implicit differentiation for good: Inverse functions.

Warmup: Calculate  $\frac{dy}{dx}$  if

1.  $e^y = xy$

2.  $\cos(y) = x + y$

Every time:

(1) Take  $\frac{d}{dx}$  of both sides.

(2) Add and subtract to get the  $\frac{dy}{dx}$  terms on one side and everything else on the other.

(3) Factor out  $\frac{dy}{dx}$  and divide both sides by its coefficient.

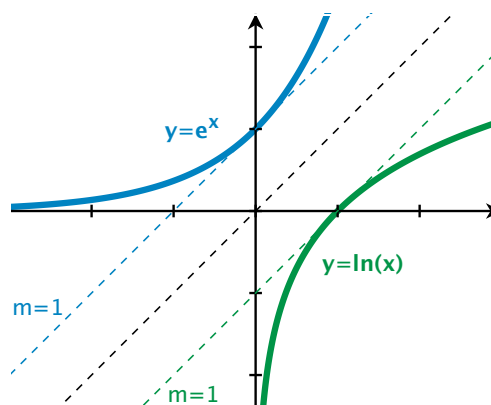
## The Derivative of $y = \ln x$

Remember:

(1)  $y = e^x$  has a slope through the point  $(0,1)$  of 1.

(2) The natural log is the *inverse* to  $e^x$ , so

$$y = \ln x \implies e^y = x$$



## The Derivative of $y = \ln x$

To find the derivative of  $\ln(x)$ , use implicit differentiation!

Rewrite

$$y = \ln x \quad \text{as} \quad e^y = x$$

Take a derivative of both sides of  $e^y = x$  to get

$$\frac{dy}{dx} e^y = 1 \quad \text{so} \quad \frac{dy}{dx} = \frac{1}{e^y}$$

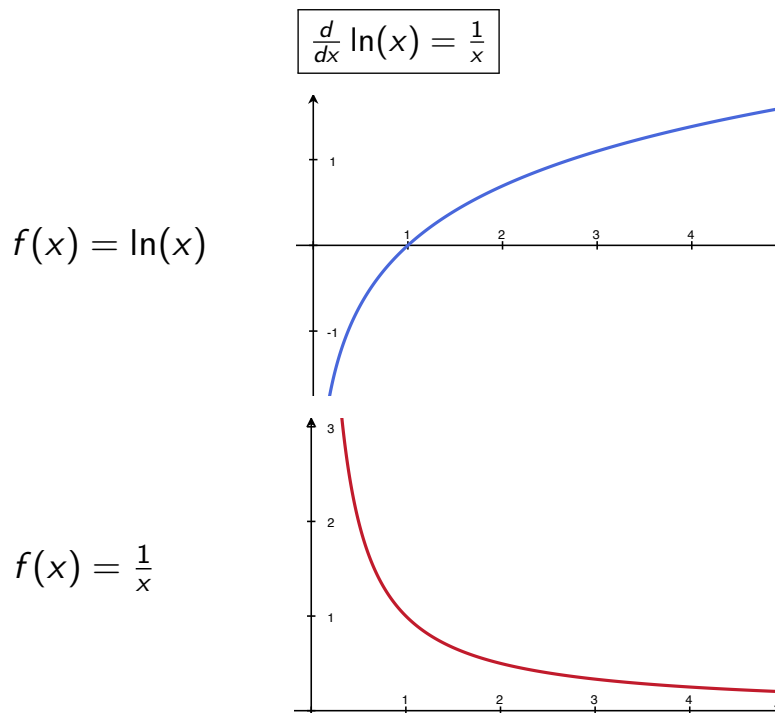
**Problem:** We asked “what is the derivative of  $\ln(x)$ ?” and got back an answer with  $y$  in it!

**Solution:** Substitute back!

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx} \ln(x) = \frac{1}{x}}$$

Does it make sense?



## Examples

Calculate

1.  $\frac{d}{dx} \ln x^2$

2.  $\frac{d}{dx} \ln(\sin(x^2))$

3.  $\frac{d}{dx} \log_3(x)$

[hint:  $\log_a x = \frac{\ln x}{\ln a}$ ]

## Quick tip: Logarithmic differentiation

Example: Calculate  $\frac{dy}{dx}$  if  $y = x^{\sin(x)}$

**Problem:** Both the base and the exponent have the variable in them! So we can't use

$$\frac{d}{dx} x^a = ax^{a-1} \quad \text{or} \quad \frac{d}{dx} a^x = \ln(a)a^x.$$

**Fix:** Take the log of both sides and use implicit differentiation:

$$\ln(y) = \ln(x^{\sin(x)}) = \sin(x) * \ln(x) \quad (\text{using } \ln(a^b) = b \ln(a))$$

Taking the derivative of both sides gives

$$\frac{1}{y} \frac{dy}{dx} = \cos(x) \ln(x) + \sin(x) \frac{1}{x}$$

Then solving for  $\frac{dy}{dx}$ ,

$$\frac{dy}{dx} = y \left( \cos(x) \ln(x) + \sin(x) \frac{1}{x} \right) = \boxed{x^{\sin(x)} \left( \cos(x) \ln(x) + \sin(x) \frac{1}{x} \right)}.$$

## Back to inverses

In the case where  $y = \ln(x)$ , we used the fact that  $\ln(x) = f^{-1}(x)$ , where  $f(x) = e^x$ , and got

$$\frac{d}{dx} \ln(x) = \frac{1}{e^{\ln(x)}}.$$

In general, calculating  $\frac{d}{dx} f^{-1}(x)$ :

(1) Rewrite  $y = f^{-1}(x)$  as  $f(y) = x$ .

(2) Use implicit differentiation:

$$f'(y) * \frac{dy}{dx} = 1 \quad \text{so} \quad \boxed{\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}}.$$

## Examples

Just to check, use the rule

$$\boxed{\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

to calculate

1.  $\frac{d}{dx} \ln(x)$  (the inverse of  $e^x$ )

In the notation above,  $f^{-1}(x) = \ln(x)$  and  $f(x) = e^x$ .

We'll also need  $f'(x) = e^x$ . So

$$\boxed{\frac{d}{dx} \ln(x) = \frac{1}{e^{\ln(x)}}} \quad \text{☺}$$

2.  $\frac{d}{dx} \sqrt{x}$  (the inverse of  $x^2$ )

In the notation above,  $f^{-1}(x) = \sqrt{x}$  and  $f(x) = x^2$ .

We'll also need  $f'(x) = 2x$ . So

$$\boxed{\frac{d}{dx} \sqrt{x} = \frac{1}{2 * (\sqrt{x})}} \quad \text{☺}$$

## Inverse trig functions

Two notations:

$f(x)$	$f^{-1}(x)$
$\sin(x)$	$\sin^{-1}(x) = \arcsin(x)$
$\cos(x)$	$\cos^{-1}(x) = \arccos(x)$
$\tan(x)$	$\tan^{-1}(x) = \arctan(x)$
$\sec(x)$	$\sec^{-1}(x) = \operatorname{arcsec}(x)$
$\csc(x)$	$\csc^{-1}(x) = \operatorname{arccsc}(x)$
$\cot(x)$	$\cot^{-1}(x) = \operatorname{arccot}(x)$

There are lots of points we know on these functions...

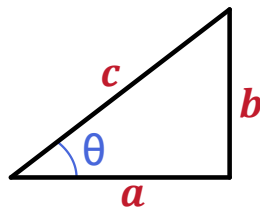
Examples:

1. Since  $\sin(\pi/2) = 1$ , we have  $\arcsin(1) = \pi/2$
2. Since  $\cos(\pi/2) = 0$ , we have  $\arccos(0) = \pi/2$

Etc...

In general:

$\arccos(-)$  takes in a ratio and spits out an angle:



$$\cos(\theta) = a/c \quad \text{so} \quad \arccos(a/c) = \theta$$

$$\sin(\theta) = b/c \quad \text{so} \quad \arcsin(b/c) = \theta$$

$$\tan(\theta) = b/a \quad \text{so} \quad \arctan(b/a) = \theta$$

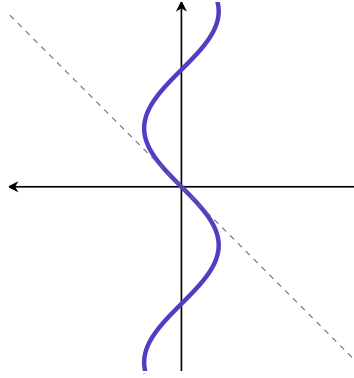
**Domain problems:**

$$\sin(0) = 0, \quad \sin(\pi) = 0, \quad \sin(2\pi) = 0, \quad \sin(3\pi) = 0, \dots$$

So which is the right answer to  $\arcsin(0)$ , really?

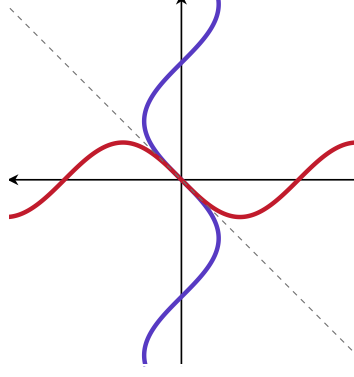
Domain/range

$$y = \sin(x)$$



Domain/range

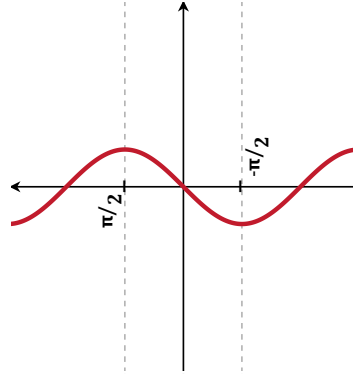
$$y = \sin(x)$$
$$y = \arcsin(x)$$



$$\text{Domain: } -1 \leq x \leq 1$$

Domain/range

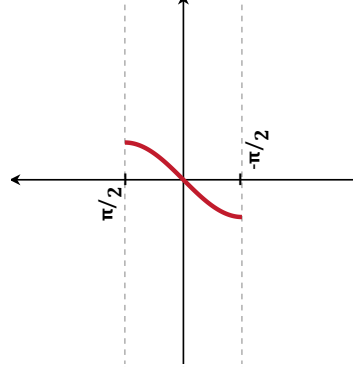
$$y = \arcsin(x)$$



$$\text{Domain: } -1 \leq x \leq 1$$

Domain/range

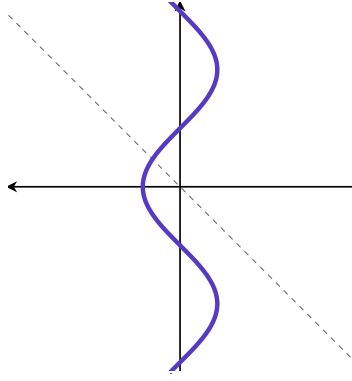
$$y = \arcsin(x)$$



$$\text{Range: } -\pi/2 \leq y \leq \pi/2$$

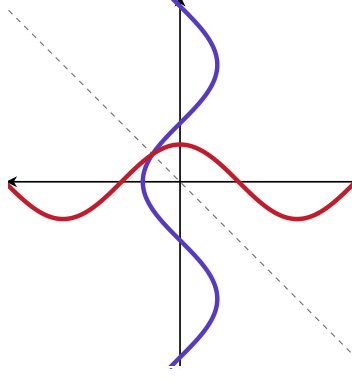
Domain/range

$$y = \cos(x)$$



Domain/range

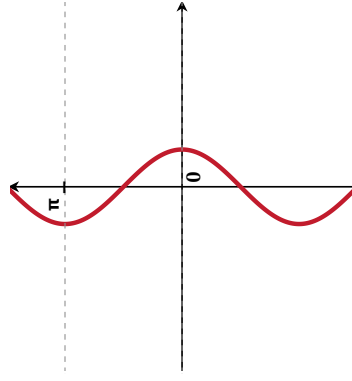
$$y = \cos(x)$$
$$y = \arccos(x)$$



Domain:  $-1 \leq x \leq 1$

Domain/range

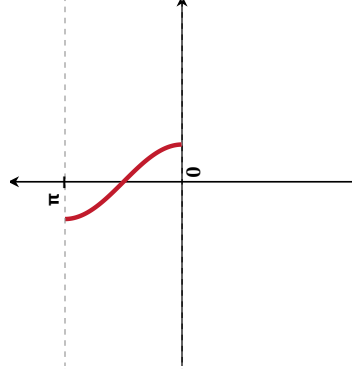
$$y = \arccos(x)$$



Domain:  $-1 \leq x \leq 1$

Domain/range

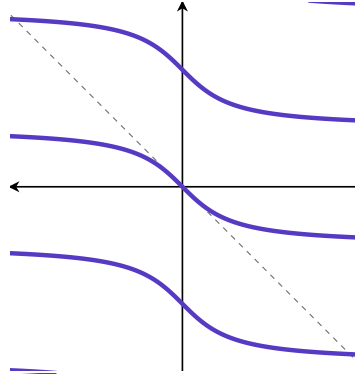
$$y = \arccos(x)$$



Domain:  $-1 \leq x \leq 1$     Range:  $0 \leq y \leq \pi$

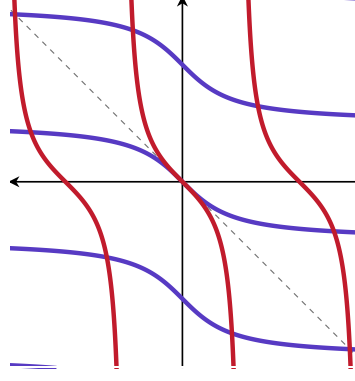
Domain/range

$$y = \tan(x)$$



Domain/range

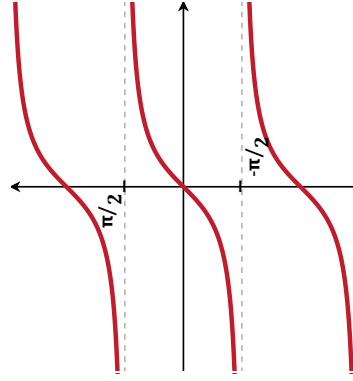
$$y = \tan(x)$$
$$y = \arctan(x)$$



Domain:  $-\infty \leq x \leq \infty$

Domain/range

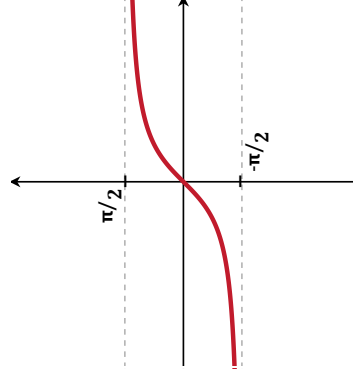
$$y = \arctan(x)$$



Domain:  $-\infty \leq x \leq \infty$

Domain/range

$$y = \arctan(x)$$

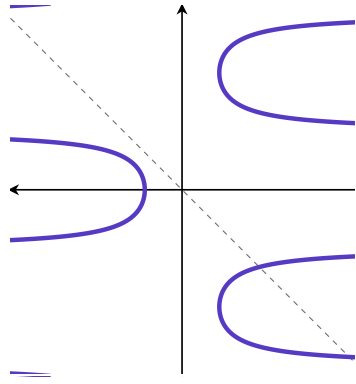


Domain:  $-\infty \leq x \leq \infty$       Range:  $-\pi/2 < y < \pi/2$



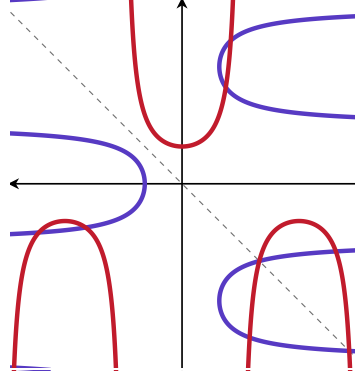
Domain/range

$$y = \sec(x)$$



Domain/range

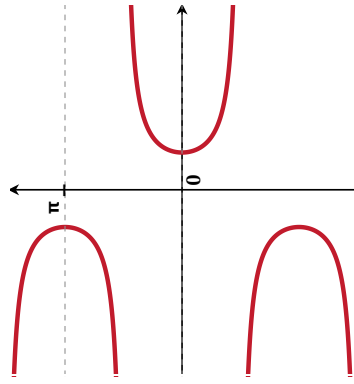
$$y = \sec(x)$$
$$y = \operatorname{arcsec}(x)$$



Domain:  $x \leq -1$  and  $1 \leq x$

Domain/range

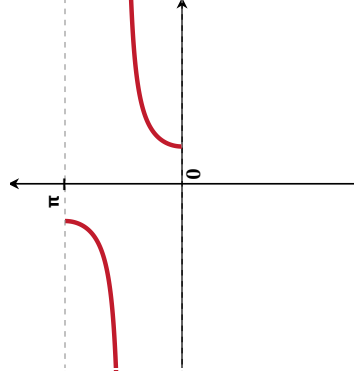
$$y = \operatorname{arcsec}(x)$$



Domain:  $x \leq -1$  and  $1 \leq x$

Domain/range

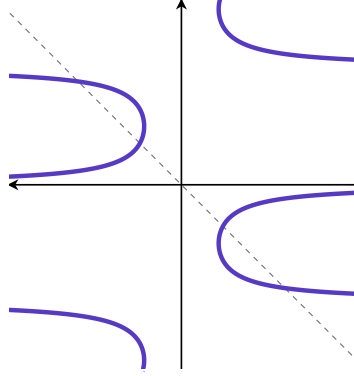
$$y = \operatorname{arcsec}(x)$$



Domain:  $x \leq -1$  and  $1 \leq x$       Range:  $0 \leq y \leq \pi$

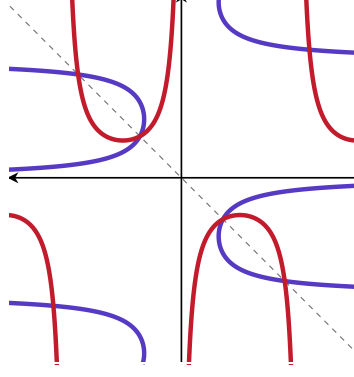
Domain/range

$$y = \csc(x)$$



Domain/range

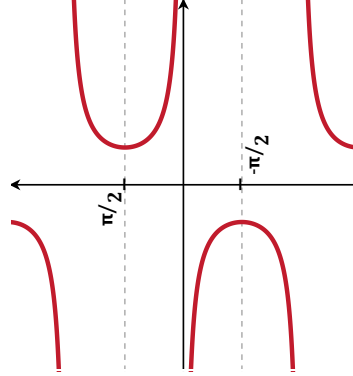
$$y = \csc(x)$$
$$y = \operatorname{arccsc}(x)$$



Domain:  $x \leq -1$  and  $1 \leq x$

Domain/range

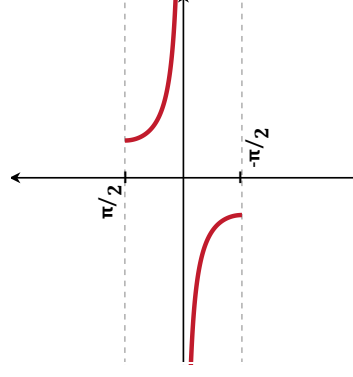
$$y = \operatorname{arccsc}(x)$$



Domain:  $x \leq -1$  and  $1 \leq x$

Domain/range

$$y = \operatorname{arccsc}(x)$$

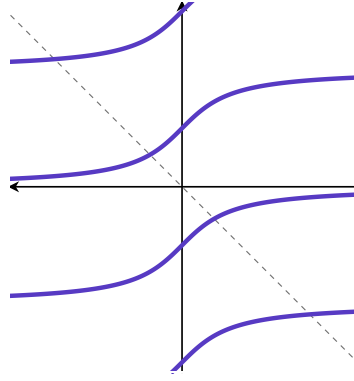


Domain:  $x \leq -1$  and  $1 \leq x$

Range:  $-\pi/2 \leq y \leq \pi/2$

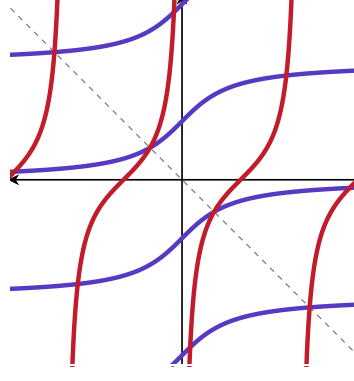
Domain/range

$$y = \cot(x)$$



Domain/range

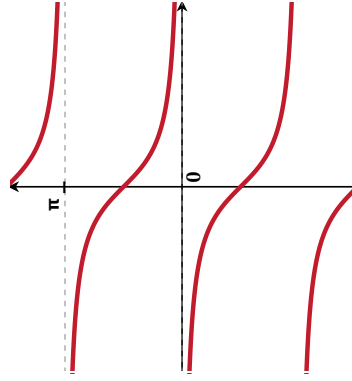
$$y = \cot(x)$$
$$y = \operatorname{arccot}(x)$$



Domain:  $-\infty \leq x \leq \infty$

Domain/range

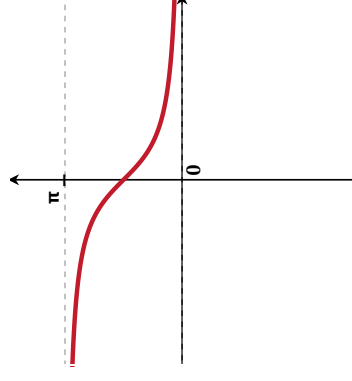
$$y = \operatorname{arccot}(x)$$



Domain:  $-\infty \leq x \leq \infty$

Domain/range

$$y = \operatorname{arccot}(x)$$



Domain:  $-\infty \leq x \leq \infty$

Range:  $0 < y < \pi$

## Back to Derivatives

Use implicit differentiation to calculate the derivatives of

1.  $\arcsin(x)$

2.  $\arctan(x)$

Use the rule

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

to check your answers, and then to calculate the derivatives of the other inverse trig functions:

1.  $\frac{d}{dx} \arccos(x)$

2.  $\frac{d}{dx} \operatorname{arcsec}(x)$

3.  $\frac{d}{dx} \operatorname{arccsc}(x)$

4.  $\frac{d}{dx} \operatorname{arccot}(x)$

Recall:

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x) \tan(x)$
$\csc(x)$	$-\csc(x) \cot(x)$
$\cot(x)$	$-\csc^2(x)$



## Using implicit differentiation to calculate $\frac{d}{dx} \arcsin(x)$

$$\text{If } y = \arcsin(x) \text{ then } x = \sin(y).$$

Take  $\frac{d}{dx}$  of both sides of  $x = \sin(y)$ :

$$\text{Left hand side: } \frac{d}{dx} x = 1$$

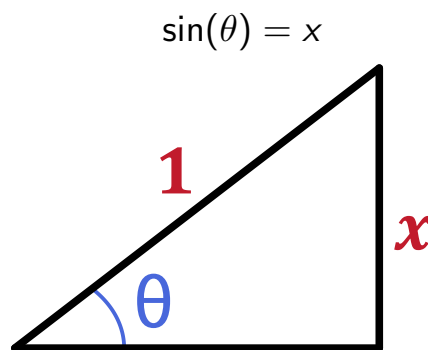
$$\text{Right hand side: } \frac{d}{dx} \sin(y) = \cos(y) * \frac{dy}{dx} = \cos(\arcsin(x)) * \frac{dy}{dx}$$

So

$$\frac{dy}{dx} = \frac{1}{\cos(\arcsin(x))}.$$

### Simplifying $\cos(\arcsin(x))$

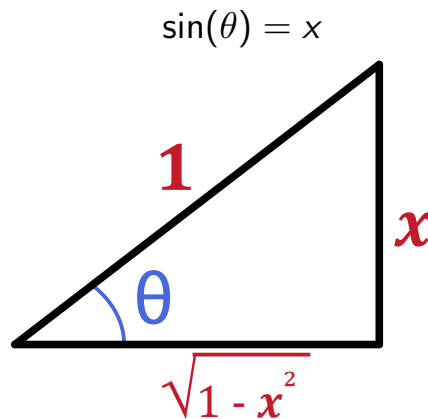
Call  $\arcsin(x) = \theta$ .



Key: This is a simple triangle to write down whose angle  $\theta$  has  $\sin(\theta) = x$

## Simplifying $\cos(\arcsin(x))$

Call  $\arcsin(x) = \theta$ .



So  $\cos(\arcsin(x)) = \sqrt{1-x^2}$

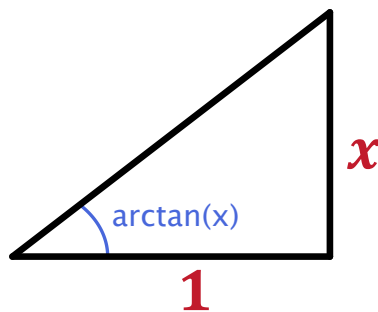
So  $\frac{d}{dx} \arcsin(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$ .

## Calculating $\frac{d}{dx} \arctan(x)$ .

We found that

$$\frac{d}{dx} \arctan(x) = \frac{1}{\sec^2(\arctan(x))} = \left( \frac{1}{\sec(\arctan(x))} \right)^2$$

Simplify this expression using



To simplify the rest, use the triangles

