Using implicit differentiation for good: Inverse functions.

Warmup: Calculate $\frac{dy}{dx}$ if

1. $e^y = xy$

   Take $\frac{d}{dx}$ of both sides to find $e^y \frac{dy}{dx} = x\frac{dy}{dx} + y$.

   So

$$y = e^y \frac{dy}{dx} - x\frac{dy}{dx} = \frac{dy}{dx}(e^y - x), \quad \text{implying} \quad \boxed{\frac{dy}{dx} = \frac{y}{e^y - x}}$$

2. $\cos(y) = x + y$

   Take $\frac{d}{dx}$ as before: $-\sin(y)\frac{dy}{dx} = 1 + \frac{dy}{dx}$. So

$$\frac{dy}{dx}(\sin(y) + 1) = -1, \quad \text{and so} \quad \boxed{\frac{dy}{dx} = \frac{-1}{\sin(y) + 1}}$$

Every time:

(1) Take $\frac{d}{dx}$ of both sides.

(2) Add and subtract to get the $\frac{dy}{dx}$ terms on one side and everything else on the other.

(3) Factor out $\frac{dy}{dx}$ and divide both sides by its coefficient.

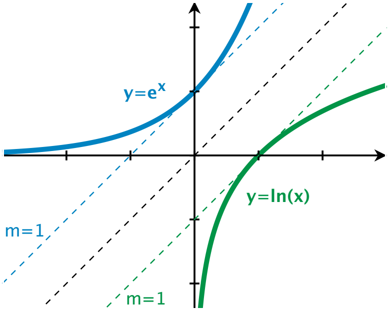
The Derivative of $y = \ln x$

Remember:

(1) $y = e^x$ has a slope through the point (0,1) of 1.

(2) The natural log is the *inverse* to e^x , so

$$y = \ln x \quad \Longrightarrow \quad e^y = x$$



The Derivative of $y = \ln x$

To find the derivative of $\ln(x)$, use implicit differentiation!
Rewrite

$$y = \ln x \quad \text{as} \quad e^y = x$$

Take a derivative of both sides of $e^y = x$ to get

$$\frac{dy}{dx} e^y = 1 \qquad \text{so} \qquad \frac{dy}{dx} = \frac{1}{e^y}$$

Problem: We asked "what is the derivative of $\ln(x)$?" and got back and answer with y in it!
Solution: Substitute back!

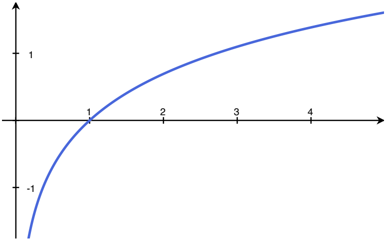
$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx} \ln(x) = \frac{1}{x}}$$

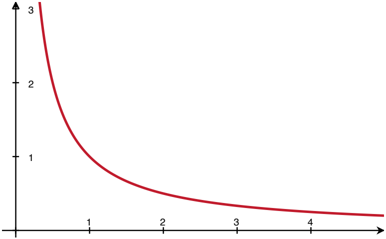
Does it make sense?

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

$$f(x) = \ln(x)$$



$$f(x) = \frac{1}{x}$$



Examples

Calculate

1. $\frac{d}{dx}\ln x^2 = \frac{2x}{x^2} = \frac{2}{x}$

2. $\frac{d}{dx}\ln(\sin(x^2)) = \frac{2x\cos(x^2)}{\sin(x^2)}$

3. $\frac{d}{dx}\log_3(x) = \frac{1}{x\ln(3)}$

   [hint: $\log_a x = \frac{\ln x}{\ln a}$]

Notice, every time:

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$$

1. $\frac{d}{dx} \ln(x^2) = \frac{1}{x^2} \cdot \frac{d}{dx} x^2 = \frac{2x}{x^2} = \frac{2}{x}$

$\hookrightarrow \frac{d}{dx} \ln(x^2) = \frac{d}{dx} 2\ln(x) = \frac{2}{x}$

yay!

2. $\frac{d}{dx} \ln(\sin(x^2)) = \frac{1}{\sin(x^2)} \cdot \frac{d}{dx} \sin(x^2)$

$$= \frac{2x\cos(x)}{\sin(x^2)}$$

3. $\frac{d}{dx} \log_a(x)$: two ways

Ⓐ $y = \log_a(x)$

$\Updownarrow$

$a^y = x$

$\downarrow d/dx$

$a^y \cdot \ln(a) \cdot \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{a^y \ln(a)} = \frac{1}{x\ln(a)}$

Ⓑ $\log_a(x) = \frac{1}{\ln(a)} \cdot \ln(x)$

so $\frac{d}{dx} \log_a(x) = \frac{1}{\ln(a)} \cdot \frac{1}{x}$.

yay!

Quick tip: Logarithmic differentiation

Example: Calculate $\frac{dy}{dx}$ if $y = x^{\sin(x)}$

Problem: Both the base and the exponent have the variable in them! So we can't use

$$\frac{d}{dx}x^a = ax^{a-1} \qquad \text{or} \qquad \frac{d}{dx}a^x = \ln(a)a^x.$$

Fix: Take the log of both sides and use implicit differentiation:

$$\ln(y) = \ln(x^{\sin(x)}) = \sin(x) * \ln(x) \qquad \text{(using } \ln(a^b) = b\ln(a)\text{)}$$

Taking the derivative of both sides gives

$$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(x) + \sin(x)\frac{1}{x}$$

Then solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = y\left(\cos(x)\ln(x) + \sin(x)\frac{1}{x}\right) = \boxed{x^{\sin(x)}\left(\cos(x)\ln(x) + \sin(x)\frac{1}{x}\right)}.$$

Back to inverses

In the case where $y = \ln(x)$, we used the fact that $\ln(x) = f^{-1}(x)$, where $f(x) = e^x$, and got

$$\frac{d}{dx}\ln(x) = \frac{1}{e^{\ln(x)}}.$$

In general, calculating $\frac{d}{dx}f^{-1}(x)$:

(1) Rewrite $y = f^{-1}(x)$ as $f(y) = x$.

(2) Use implicit differentiation:

$$f'(y) * \frac{dy}{dx} = 1 \qquad \text{so} \qquad \boxed{\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}}.$$

Examples

Just to check, use the rule

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

to calculate

1. $\frac{d}{dx} \ln(x)$ (the inverse of e^x)
   In the notation above, $f^{-1}(x) = \ln(x)$ and $f(x) = e^x$.
   We'll also need $f'(x) = e^x$. So

   $$\frac{d}{dx} \ln(x) = \frac{1}{e^{\ln(x)}}$$ ☺

2. $\frac{d}{dx} \sqrt{x}$ (the inverse of x^2)
   In the notation above, $f^{-1}(x) = \sqrt{x}$ and $f(x) = x^2$.
   We'll also need $f'(x) = 2x$. So

   $$\frac{d}{dx} \sqrt{x} = \frac{1}{2 * (\sqrt{x})}$$ ☺

Inverse trig functions

Two notations:

$f(x)$	$f^{-1}(x)$
$\sin(x)$	$\sin^{-1}(x) = \arcsin(x)$
$\cos(x)$	$\cos^{-1}(x) = \arccos(x)$
$\tan(x)$	$\tan^{-1}(x) = \arctan(x)$
$\sec(x)$	$\sec^{-1}(x) = \text{arcsec}(x)$
$\csc(x)$	$\csc^{-1}(x) = \text{arccsc}(x)$
$\cot(x)$	$\cot^{-1}(x) = \text{arccot}(x)$

There are lots of points we know on these functions...

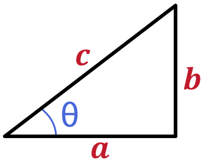
Examples:

1. Since $\sin(\pi/2) = 1$, we have $\arcsin(1) = \pi/2$
2. Since $\cos(\pi/2) = 0$, we have $\arccos(0) = \pi/2$

Etc...

In general:

arc___(-) takes in a ratio and spits out an angle:



$\cos(\theta) = a/c$ so $\arccos(a/c) = \theta$

$\sin(\theta) = b/c$ so $\arcsin(b/c) = \theta$

$\tan(\theta) = b/a$ so $\arctan(b/a) = \theta$

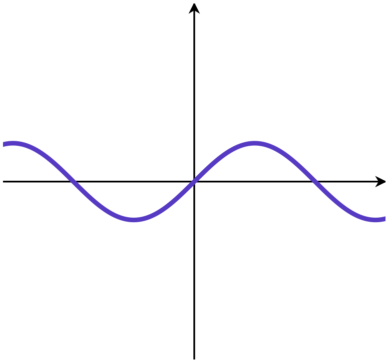
Domain problems:

$\sin(0) = 0, \quad \sin(\pi) = 0, \quad \sin(2\pi) = 0, \quad \sin(3\pi) = 0, \ldots$

So which is the right answer to arcsin(0), really?

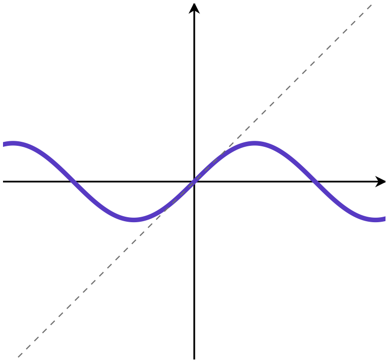
Domain/range

$$y = \sin(x)$$



Domain/range

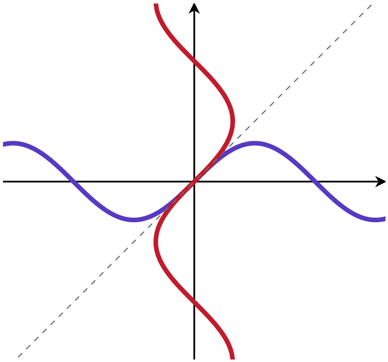
$$y = \sin(x)$$



Domain/range

$y = \sin(x)$

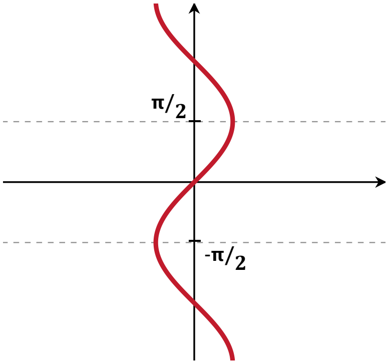
$y = \arcsin(x)$



Domain: $-1 \leq x \leq 1$

Domain/range

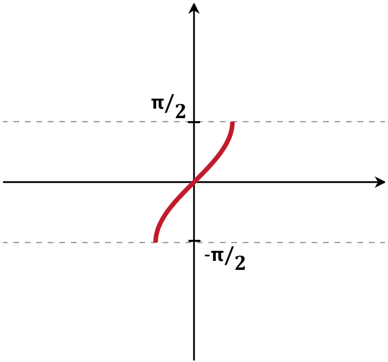
$y = \arcsin(x)$



Domain: $-1 \leq x \leq 1$

Domain/range

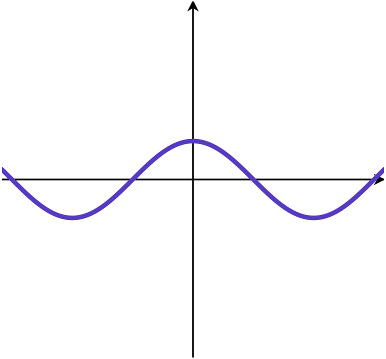
$y = \arcsin(x)$



Domain: $-1 \leq x \leq 1$ Range: $-\pi/2 \leq y \leq \pi/2$

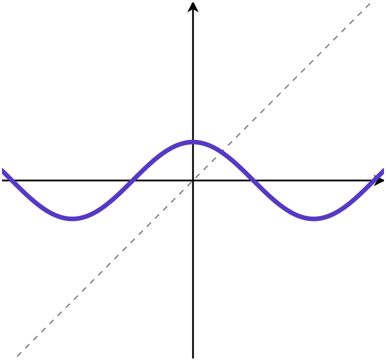
Domain/range

$$y = \cos(x)$$



Domain/range

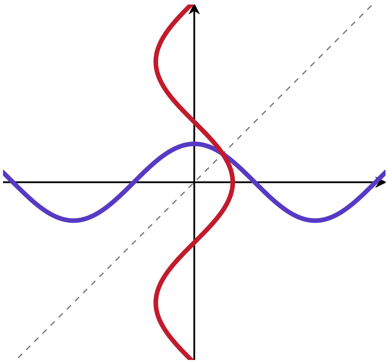
$y = \cos(x)$



Domain/range

$y = \cos(x)$

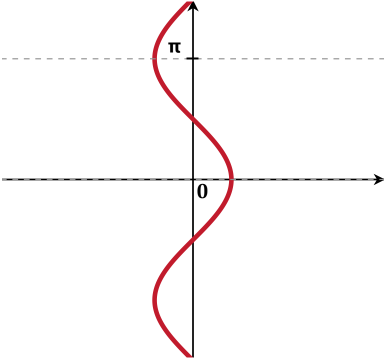
$y = \arccos(x)$



Domain: $-1 \leq x \leq 1$

Domain/range

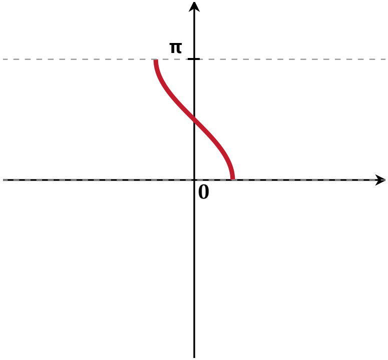
$y = \arccos(x)$



Domain: $-1 \leq x \leq 1$

Domain/range

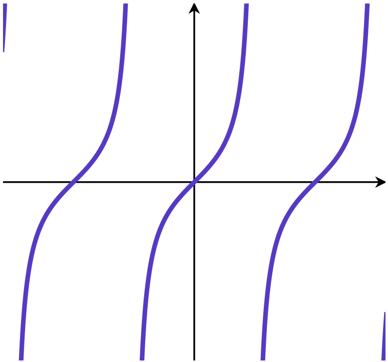
$y = \text{arccos}(x)$



Domain: $-1 \leq x \leq 1$ Range: $0 \leq y \leq \pi$

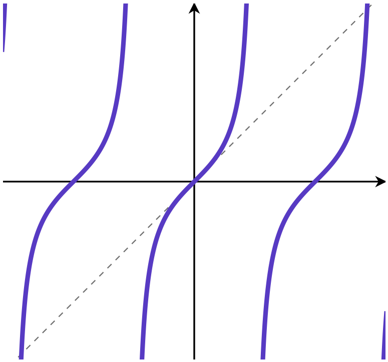
Domain/range

$y = \tan(x)$



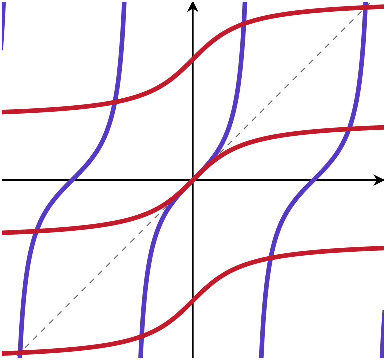
Domain/range

$$y = \tan(x)$$



Domain/range

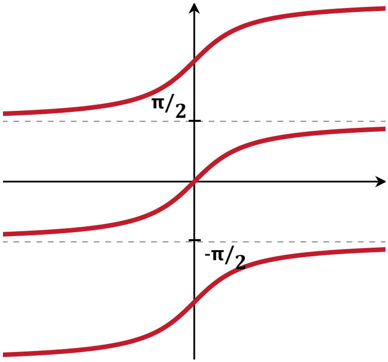
$y = \tan(x)$
$y = \arctan(x)$



Domain: $-\infty \leq x \leq \infty$

Domain/range

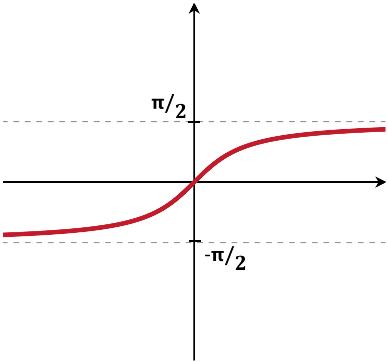
$y = \arctan(x)$



Domain: $-\infty \leq x \leq \infty$

Domain/range

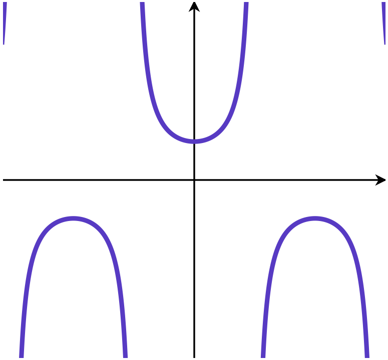
$y = \arctan(x)$



Domain: $-\infty \le x \le \infty$ Range: $-\pi/2 < y < \pi/2$

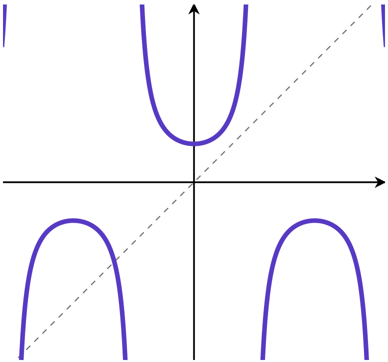
Domain/range

$y = \sec(x)$



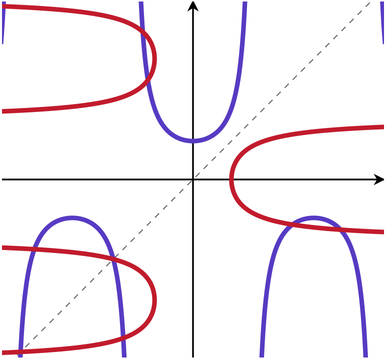
Domain/range

$y = \sec(x)$



Domain/range

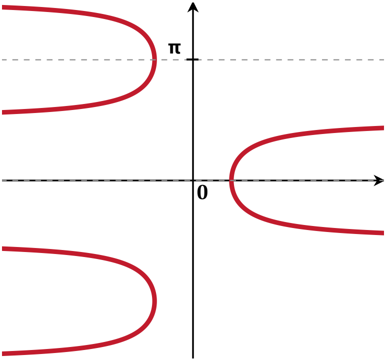
$y = \sec(x)$
$y = \text{arcsec}(x)$



Domain: $x \leq -1$ and $1 \leq x$

Domain/range

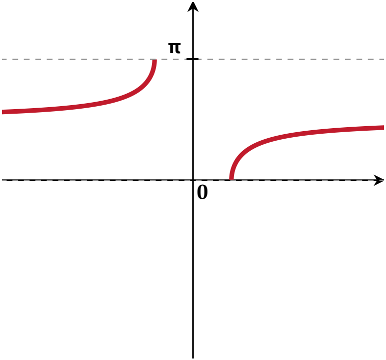
$y = \mathrm{arcsec}(x)$



Domain: $x \leq -1$ and $1 \leq x$

Domain/range

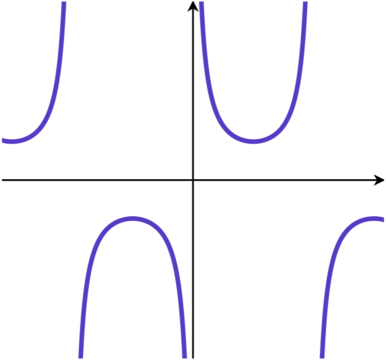
$y = \text{arcsec}(x)$



Domain: $x \leq -1$ and $1 \leq x$ Range: $0 \leq y \leq \pi$

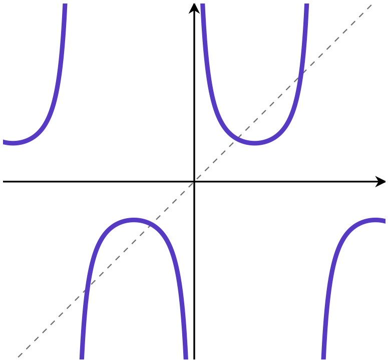
Domain/range

$y = \csc(x)$



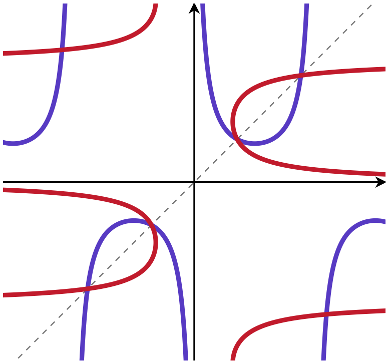
Domain/range

$$y = \csc(x)$$



Domain/range

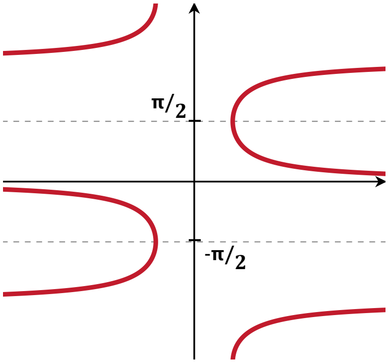
$y = \csc(x)$
$y = \mathrm{arccsc}(x)$



Domain: $x \leq -1$ and $1 \leq x$

Domain/range

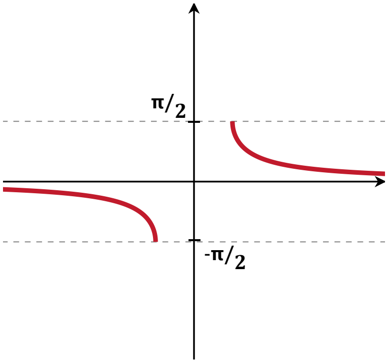
$y = \text{arccsc}(x)$



Domain: $x \leq -1$ and $1 \leq x$

Domain/range

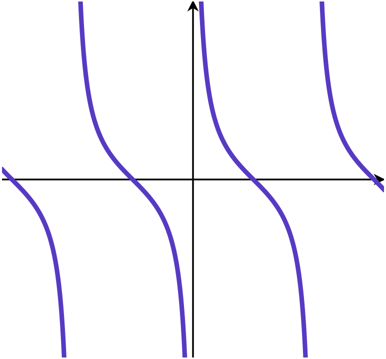
$y = \text{arccsc}(x)$



Domain: $x \leq -1$ and $1 \leq x$ Range: $-\pi/2 \leq y \leq \pi/2$

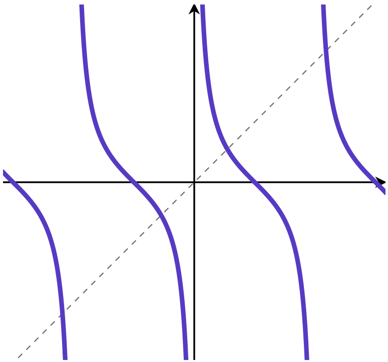
Domain/range

$$y = \cot(x)$$



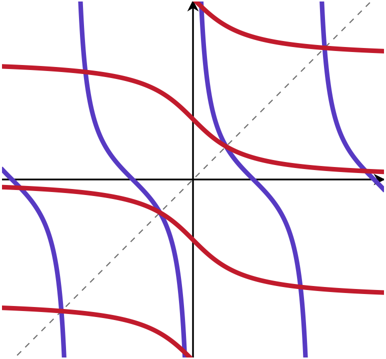
Domain/range

$$y = \cot(x)$$



Domain/range

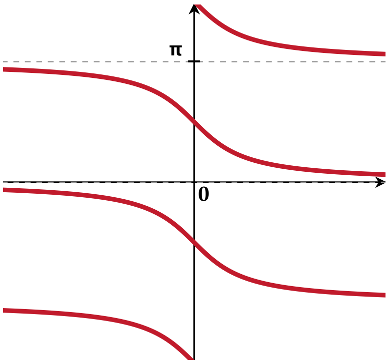
$y = \cot(x)$
$y = \text{arccot}(x)$



Domain: $-\infty \leq x \leq \infty$

Domain/range

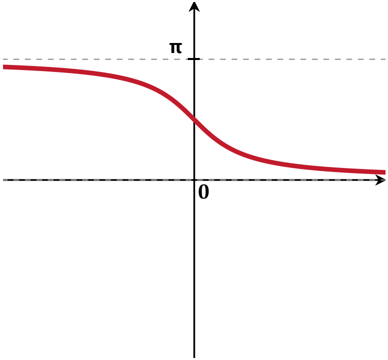
$y = \mathrm{arccot}(x)$



Domain: $-\infty \leq x \leq \infty$

Domain/range

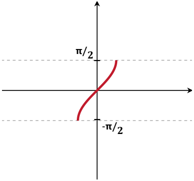
$y = \text{arccot}(x)$



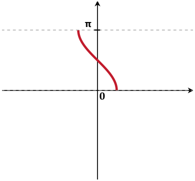
Domain: $-\infty \leq x \leq \infty$ Range: $0 < y < \pi$

Graphs

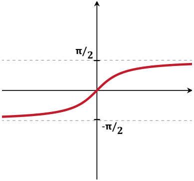
$\arcsin(x)$



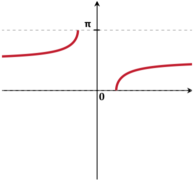
$\arccos(x)$



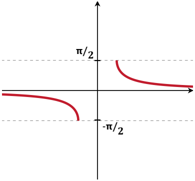
$\arctan(x)$



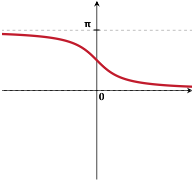
$\operatorname{arcsec}(x)$



$\operatorname{arccsc}(x)$



$\operatorname{arccot}(x)$



Back to Derivatives

Use implicit differentiation to calculate the derivatives of

1. $\arcsin(x) = \frac{1}{\cos(\arcsin(x))}$
2. $\arctan(x) = \frac{1}{\sec^2(\arctan(x))}$

Use the rule

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

to check your answers, and then to calculate the derivatives of the other inverse trig functions:

1. $\frac{d}{dx} \arccos(x) = -\frac{1}{\sin(\arccos(x)}$
2. $\frac{d}{dx} \mathrm{arcsec}(x) = \frac{1}{\sec(\mathrm{arcsec}(x)) \tan(\mathrm{arcsec}(x))}$
3. $\frac{d}{dx} \mathrm{arccsc}(x) = -\frac{1}{\csc(\mathrm{arccsc}(x)) \cot(\mathrm{arccsc}(x))}$
4. $\frac{d}{dx} \mathrm{arccot}(x) = -\frac{1}{\csc^2(\mathrm{arccot}(x))}$

Using implicit differentiation to calculate $\frac{d}{dx}\arcsin(x)$

$$\boxed{\text{If } y = \arcsin(x) \text{ then } x = \sin(y).}$$

Take $\frac{d}{dx}$ of both sides of $x = \sin(y)$:

Left hand side: $$\frac{d}{dx}x = 1$$

Right hand side: $$\frac{d}{dx}\sin(y) = \cos(y) * \frac{dy}{dx} = \cos(\arcsin(x)) * \frac{dy}{dx}$$

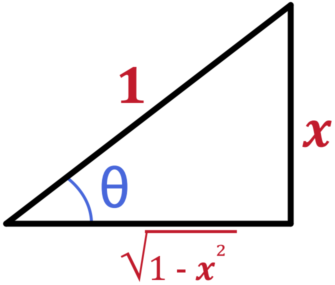
So

$$\boxed{\frac{dy}{dx} = \frac{1}{\cos(\arcsin(x))}.}$$

Simplifying $\cos(\arcsin(x))$

Call $\arcsin(x) = \theta$.

$\sin(\theta) = x$



So $\quad \cos(\arcsin(x)) = \sqrt{1 - x^2}$

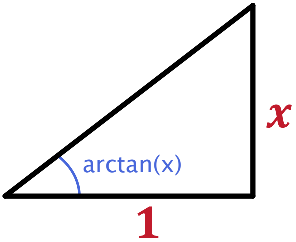
So $\quad \dfrac{d}{dx}\arcsin(x) = \dfrac{1}{\cos(\arcsin(x))} = \dfrac{1}{\sqrt{1-x^2}}.$

Calculating $\frac{d}{dx}\arctan(x)$.

We found that

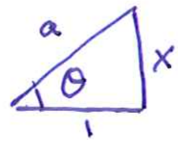
$$\frac{d}{dx}\arctan(x) = \frac{1}{\sec^2(\arctan(x))} = \left(\frac{1}{\sec(\arctan(x))}\right)^2$$

Simplify this expression using



$$\frac{dy}{dx} = \left(\frac{1}{\sec(\arctan(x))}\right)^2 = \frac{1}{1+x^2}$$

$$(\star)\ \frac{d}{dx}\arctan(x) = \left(\frac{1}{\sec(\underbrace{\arctan(x)}_{\theta})}\right)^2 = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \boxed{\frac{1}{1+x^2}}$$



$\sec(\theta) = a/1$

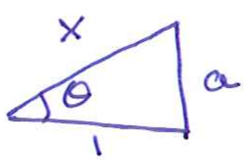
$= \boxed{\sqrt{1+x^2}}$

$a^2 = 1^2 + x^2$

so $a = \sqrt{1+x^2}$

$$(\star)\ \frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{\sec(\operatorname{arcsec}(x))\tan(\underbrace{\operatorname{arcsec}(x)}_{\theta})} = \boxed{\frac{1}{x\sqrt{x^2-1}}}$$

$\begin{cases} \frac{1}{x\sqrt{\ }} & x>0 \\ \frac{1}{-x\cdot\sqrt{\ }} & x<0 \end{cases}$



$\tan\theta = \sqrt{x^2-1}$

$\sec\theta = x$

$1^2 + a^2 = x^2$

$a = \sqrt{x^2-1}$

uh oh!
look back
at graph...

actually $\boxed{\frac{1}{|x|\sqrt{x^2-1}}}$

$\frac{d}{dx}\cos^{-1}(x) = -\frac{d}{dx}\sin^{-1}(x)$

$\frac{d}{dx}\cot^{-1}(x) = -\frac{d}{dx}\tan^{-1}(x)$

$\frac{d}{dx}\csc^{-1}(x) = -\frac{d}{dx}\sec^{-1}(x)$

To simplify the rest, use the triangles