Warm-up

Implicit Differentiation

Suppose $f(x)$ is a differentiable function. Then in terms of x, $f(x)$ and $f'(x)$, what are

1. $\frac{d}{dx}(xf(x))$
2. $\frac{d}{dx}(f(x)+\sin(x))$
3. $\frac{d}{dx}\left(f(x)/x\right)$
4. $\frac{d}{dx}\left(f^2(x)\right)$
5. $\frac{d}{dx}\left(1+xf^4(x)\right)$
6. $\frac{d}{dx}\left(\cos\left(xe^{f(x)}\right)\right)$

Implicit functions

An implicit function is something that can we written as

$$F(x, y) = c.$$

So it might not be a function, but it is a curve in the x-y plane.

For example, $x^2 + y^2 = 1$.

Question:

What is the slope of the tangent line to the unit circle at $x = \frac{\sqrt{2}}{2}$?

Using calculus so far...

Solve for y:

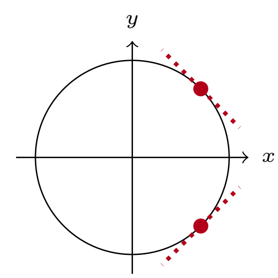
$$y = \pm\sqrt{1 - x^2}$$

Take a derivative:

$$\frac{dy}{dx} = -\frac{\pm 2x}{2\sqrt{1-x^2}}$$

Plug in the point:

$$\left.\frac{dy}{dx}\right|_{x=\frac{\sqrt{2}}{2}} = -\frac{\pm\sqrt{2}}{2\sqrt{1-1/2}} = \mp 1$$



But which is which??

Implicit differentiation

Instead, think of y as a function of x: let $y = f(x)$.

Then rewrite

$$x^2 + y^2 = 1 \quad \text{as} \quad x^2 + f^2(x) = 1.$$

(1) Take a derivative of both sides independently:

$$\frac{d}{dx}(x^2 + f^2(x)) = 2x + 2f(x) * f'(x) = 2x + 2y\frac{dy}{dx}$$

$$\frac{d}{dx}1 = 0$$

(2) So $\boxed{2x + 2y\frac{dy}{dx} = 0}$

(3) solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = -2x/2y = -x/y.}$$

(4) Plug in points:

x	y	$\frac{dy}{dx}$
$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	
$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	

Implicit differentiation

In general, given $F(x, y) = c$, calculate $\frac{dy}{dx}$ in terms of x and y.

Step 1: Take derivative of both sides versus x.
(don't forget the chain rule!)

Step 2: Set the two sides equal.

Step 3: Solve for $\frac{dy}{dx}$.

Step 4: If applicable, plug in points.

For example, calculate $\frac{dy}{dx}$ if $xy^2 = 3$.

(1) $\frac{d}{dx}xy^2 = 1 * y^2 + x * \left(2y * \frac{dy}{dx}\right)$ and $\frac{d}{dx}3 = 0$

(2) So $y^2 + 2xy\frac{dy}{dx} = 0$

(3) and so $\boxed{\frac{dy}{dx} = -y^2/2xy = \frac{-y}{2x}}$

Examples

Calculate $\frac{dy}{dx}$ if. . .

(a) $xy = 1$

(b) $y + \sin(x) = xy$

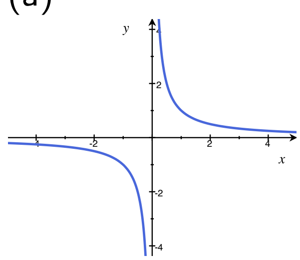
(c) $y/x = xy^4 + 1$

(d) $\cos(xe^y) = y^2$

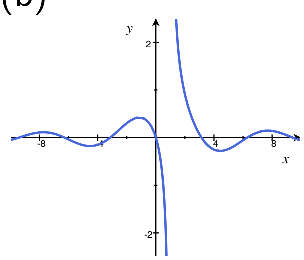
Then $\frac{dx}{dy}$ for (a) and (b) as well. To do this, now pretend that $x = f(y)$. See a pattern?

For reference, the graphs look like

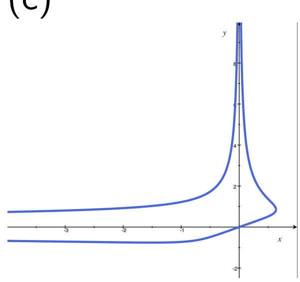
(a)



(b)



(c)



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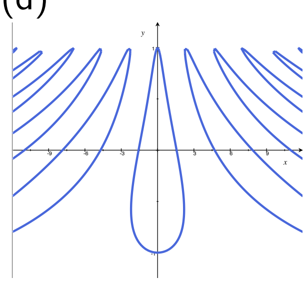


Vertical and horizontal tangent lines

We saw that if $x^2 + y^2 = 1$, then $\frac{dy}{dx} = -x/y$.

Question 1: At which points is the tangent line horizontal?
Solve $0 = \frac{dy}{dx}$:

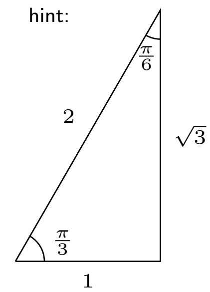
$$0 = -x/y \quad \text{if } x = 0$$

Question 2: At which points is the tangent line vertical?

Other tangent lines

We saw that if $x^2 + y^2 = 1$, then $\frac{dy}{dx} = -x/y$.

Question 3: Find the equations for all lines which are tangent to the unit circle and have slope $\sqrt{3}$.



Higher derivatives;

Calculate $\frac{d^2y}{dx^2} = y''$ if $xy^2 = 3$.

You can start from either

$$y^2 + 2xyy' = 0 \qquad \text{or} \qquad y' = -y/2x.$$

$$\begin{aligned} y'' = \frac{d}{dx}(-y/2x) &= -\frac{y' * 2x - y * 2}{(2x)^2} \\ &= -\frac{y'x - y}{2x^2} \\ &= -\frac{(-y/2x)x - y}{2x^2} = \boxed{\frac{3y}{4x^2}} \end{aligned}$$

Get your answer back into just x's and y's!