Implicit differentiation

Warm-up

Suppose $f(x)$ is a differentiable function. Then in terms of x, $f(x)$ and $f'(x)$, what are

1. $\frac{d}{dx}(xf(x)) = f(x) + xf'(x)$
2. $\frac{d}{dx}(f(x) + \sin(x)) = f'(x) + \cos(x)$
3. $\frac{d}{dx}\left(f(x)/x\right) = f'(x)/x - f(x)/x^2$
4. $\frac{d}{dx}\left(f^2(x)\right) = 2f(x)f'(x)$
5. $\frac{d}{dx}\left(1 + xf^4(x)\right) = f^4(x) + 4xf^3(x)f'(x)$
6. $\frac{d}{dx}\left(\cos\left(xe^{f(x)}\right)\right) = -\sin\left(xe^{f(x)}\right) * (1 + xf'(x)) * e^{f(x)}$

Implicit functions

An implicit function is something that can we written as

$$F(x,y)=c.$$

So it might not be a function, but it is a curve in the x-y plane.

For example, $x^2+y^2=1$.

Question:

What is the slope of the tangent line to the unit circle at $x=\frac{\sqrt{2}}{2}$?

Using calculus so far...

Solve for y:

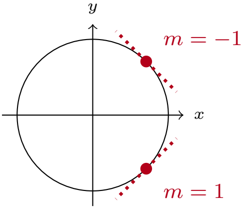
$$y=\pm\sqrt{1-x^2}$$

Take a derivative:

$$\frac{dy}{dx}=-\frac{\pm 2x}{2\sqrt{1-x^2}}$$

Plug in the point:

$$\left.\frac{dy}{dx}\right|_{x=\frac{\sqrt{2}}{2}}=-\frac{\pm\sqrt{2}}{2\sqrt{1-1/2}}=\mp 1$$



But which is which??

...we can do better.

Implicit differentiation

Instead, think of y as a function of x: let $y = f(x)$.
Then rewrite

$$x^2 + y^2 = 1 \quad \text{as} \quad x^2 + f^2(x) = 1.$$

(1) Take a derivative of both sides independently:

$$\frac{d}{dx}(x^2 + f^2(x)) = 2x + 2f(x) * f'(x) = 2x + 2y\frac{dy}{dx}$$

$$\frac{d}{dx}1 = 0$$

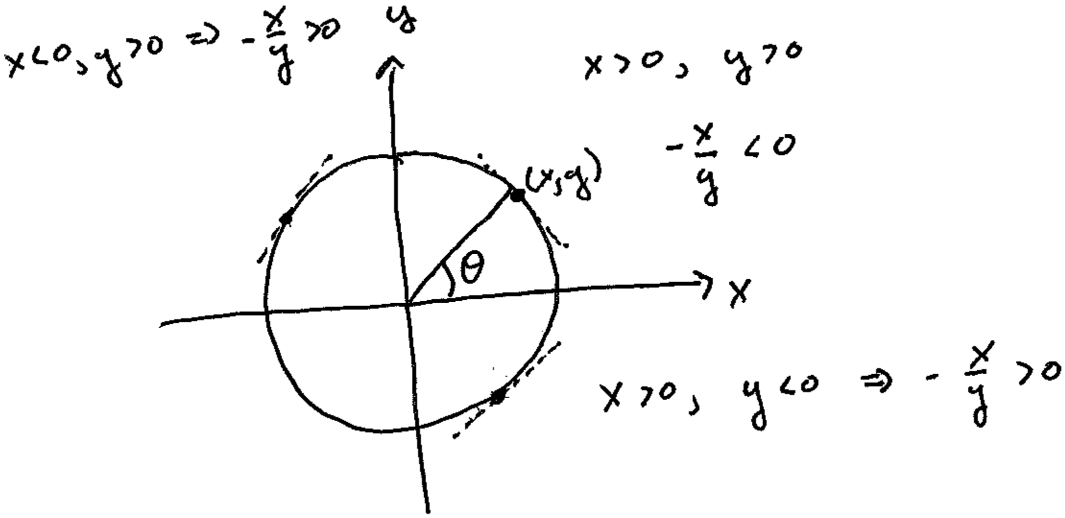
(2) So $\boxed{2x + 2y\frac{dy}{dx} = 0}$

(3) solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = -2x/2y = -x/y.}$$

(4) Plug in points:

x	y	$\frac{dy}{dx}$
$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1

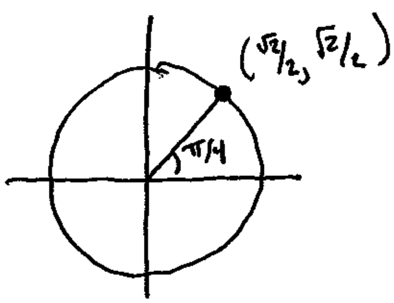


if $x = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\left(\frac{1}{\sqrt{2}}\right)^2 + y^2 = 1$ $\rightarrow y^2 = \frac{1}{2}$

$\frac{1}{2} + y^2 =$

so $y = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$



$-\frac{x}{y} = -\frac{\cos(\theta)}{\sin(\theta)} = -\cot(\theta)$

Implicit differentiation

In general, given $F(x, y) = c$, calculate $\frac{dy}{dx}$ in terms of x and y.

Step 1: Take derivative of both sides versus x.
(don't forget the chain rule!)

Step 2: Set the two sides equal.

Step 3: Solve for $\frac{dy}{dx}$.

Step 4: If applicable, plug in points.

For example, calculate $\frac{dy}{dx}$ if $xy^2 = 3$.

(1) $\frac{d}{dx}xy^2 = 1 * y^2 + x * \left(2y * \frac{dy}{dx}\right)$ and $\frac{d}{dx}3 = 0$

(2) So $y^2 + 2xy\frac{dy}{dx} = 0$

(3) and so $\boxed{\frac{dy}{dx} = -y^2/2xy = \frac{-y}{2x}}$

Examples

Calculate $\frac{dy}{dx}$ if...

(a) $xy = 1$

(b) $y + \sin(x) = xy$

(c) $y/x = xy^4 + 1$

(d) $\cos(xe^y) = y^2$

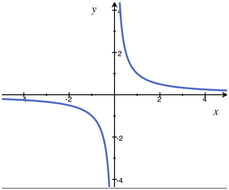
Then $\frac{dx}{dy}$ for (a) and (b) as well. To do this, now pretend that $x = f(y)$. See a pattern?

In general,

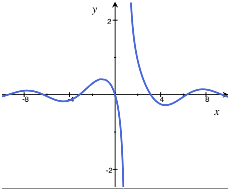
$$\boxed{\frac{dy}{dx} = 1/\left(\frac{dx}{dy}\right).}$$

For reference, the graphs look like

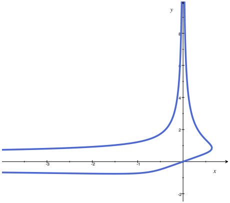
(a)



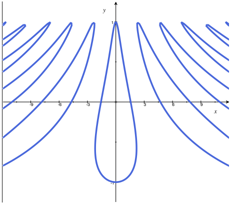
(b)



(c)



(d)



(a) $\underline{xy = 1}$:

$$\frac{d}{dx}(xy) = y + xy'$$

$$= \frac{d}{dx}(1) = 0$$

so $xy' = -y$

$$\boxed{y' = -y/x}$$

(b) $y + \sin(x) = xy$

$$\frac{d}{dx}(y + \sin(x)) = y' + \cos(x)$$

$$= \frac{d}{dx}(xy) = y + xg'$$

so $y' - xy' = y - \cos(x)$

$$y'(1 - x) =$$

so $$\boxed{y' = \frac{y - \cos(x)}{1 - x}}$$

(c) $y/x = xy^4 + 1$

$$\frac{d}{dx}(y/x) = y'/x - y/x^2$$

$$= \frac{d}{dx}(xy^4+1) = y^4 + 4xy^3y'$$

so $$y'\left(\frac{1}{x}\right) - y' \cdot 4xy^3 = \frac{y}{x^2} + y^4$$

$$y'\left(\frac{1}{x} - 4xy^3\right) =$$

so $$\boxed{y' = \frac{y/x^2 + y^4}{\frac{1}{x} - 4xy^3}}$$

(d) $\cos(xe^y) = y^2$

$$\frac{d}{dx}(\cos(xe^y)) = -\sin(xe^y)\cdot(e^y + xe^y y')$$

$$= -\sin(xe^y)e^y - y' \sin(xe^y)\cdot xe^y$$

$$= \frac{d}{dx} y^2 = 2yy'$$

so $$\boxed{y' = -\frac{\sin(xe^y)\cdot e^y}{2y + \sin(xe^y)\cdot xe^y}}$$

(a) versus y

$$\frac{d}{dy}(xy) = \frac{dx}{dy}\cdot y + x\cdot 1$$

$$= \frac{d}{dy}(1) = 0$$

$$\frac{dx}{dy}\cdot y + x = 0 \qquad \text{so} \quad \boxed{\frac{dx}{dy} = -x/y}$$

(b) $y + \sin(x) = xy$ versus y:

$$\frac{d}{dy}(y+\sin(x)) = 1 + \cos(x)\cdot\frac{dx}{dy}$$

$$= \frac{d}{dy}(xy) = \frac{dx}{dy}\cdot y + x$$

$$\frac{dx}{dy}\cdot y - \frac{dx}{dy}\cdot\cos(x) = 1 - x$$

$$\frac{dx}{dy}(y - \cos(x)) =$$

$$\boxed{\frac{dx}{dy} = \frac{1-x}{y-\cos(x)}} = \frac{x-1}{\cos(x)-y}$$

Vertical and horizontal tangent lines

We saw that if $x^2 + y^2 = 1$, then $\frac{dy}{dx} = -x/y$.

Question 1: At which points is the tangent line horizontal?
Solve $0 = \frac{dy}{dx}$:

$$0 = -x/y \quad \text{if } x = 0$$

To get the y-coordinate, plug back in and solve:

$$0^2 + y^2 = 1, \quad \text{so } y = \pm 1.$$

Answer: $\boxed{\text{at } (0,1) \text{ and } (0,-1).}$

Question 2: At which points is the tangent line vertical?
Solve: where is $\frac{dy}{dx}$ approaching $\pm\infty$? At $y = 0$!
To get the x-coordinate, plug back in and solve:

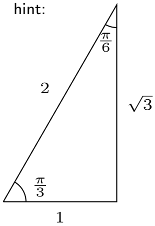
$$x^2 + 0^2 = 1, \quad \text{so } x = \pm 1.$$

Answer: $\boxed{\text{at } (1,0) \text{ and } (-1,0).}$

Other tangent lines

We saw that if $x^2 + y^2 = 1$, then $\frac{dy}{dx} = -x/y$.

Question 3: Find the equations for all lines which are tangent to the unit circle and have slope $\sqrt{3}$.



Solution: Solve $\sqrt{3} = \frac{dy}{dx} = -x/y$.
This happens when

$$\boxed{\cos(\theta)/\sin(\theta) = x/y = -\sqrt{3}.}$$

So $\theta = -\pi/6$ or $\theta = \pi - \pi/6 = 5\pi/6$.

At $\theta = -\pi/6$, we have $x = \sqrt{3}/2, y = -1/2$, so the tangent line is

$$\boxed{y = \sqrt{3}(x - \sqrt{3}/2) - 1/2}$$

At $\theta = 5\pi/6$, we have $x = -\sqrt{3}/2, y = 1/2$, so the tangent line is

$$\boxed{y = \sqrt{3}(x + \sqrt{3}/2) + 1/2}$$

Higher derivatives;

Calculate $\frac{d^2y}{dx^2} = y''$ if $xy^2 = 3$.

You can start from either

$$y^2 + 2xyy' = 0 \qquad \text{or} \qquad y' = -y/2x.$$

$$\begin{aligned} y'' = \frac{d}{dx}(-y/2x) &= -\frac{y' * 2x - y * 2}{(2x)^2} \\ &= -\frac{y'x - y}{2x^2} \\ &= -\frac{(-y/2x)x - y}{2x^2} = \boxed{\frac{3y}{4x^2}} \end{aligned}$$

Get your answer back into just x's and y's!