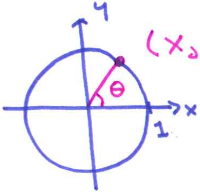
Derivatives of the Trig and Exponential Functions

Trig Identities:



$(x, y) = (\cos\theta, \sin\theta)$

Resulting identities:

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\cos^2(\theta) + \sin^2\theta = 1$$

Other useful identities:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

The derivative of sine

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}
\end{aligned}
$$

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&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h}
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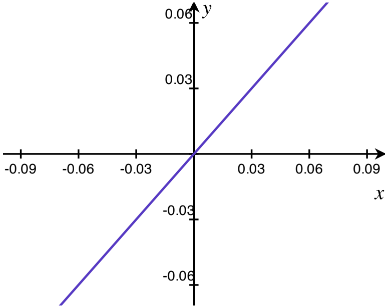
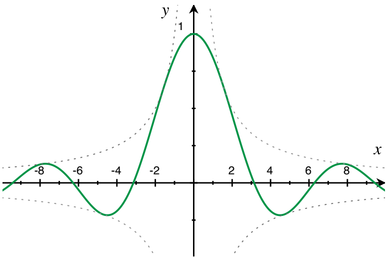
\end{aligned}$$

Recall: $\cos(0) = 1$ and $\sin(0) = 0$

Near $x = 0$, $\sin(x) \approx x$:

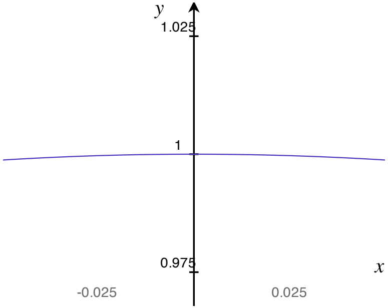


Graph of $\frac{\sin(x)}{x}$:

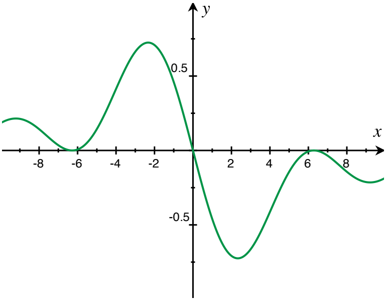


$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Near $x = 0$, $\cos(x) \approx 1$:



Graph of $\frac{\cos(x)-1}{x}$:



$$\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0$$

The derivative of sine

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}
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&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \\
&= \sin(x)*0+\cos(x)*1 \\
&= \boxed{\cos(x)}
\end{aligned}$$

The derivative of cosine

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$

The derivative of cosine

$$\begin{aligned}\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}\\ &= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}\\ &= \lim_{h\to 0}\frac{\cos(x)(\cos(h)-1)-\sin(x)\sin(h)}{h}\end{aligned}$$

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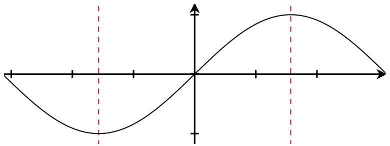
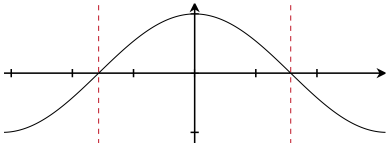
&= \cos(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}-\sin(x)\lim_{h\to 0}\frac{\sin(h)}{h} \\
&= \cos(x)*0-\sin(x)*1 \\
&= \boxed{-\sin(x)}
\end{aligned}
$$

Does it make sense?

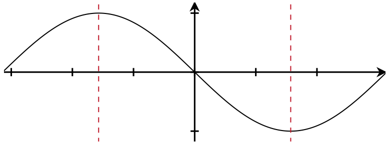
$y = \sin(x)$:



$y = \cos(x)$:



$y = -\sin(x)$:



Examples

Calculate...

1. $\frac{d}{dx}\sin(2x) = \boxed{2 * \cos(2x)}$

2. $\frac{d}{dx}\cos(3x + \sqrt{x})$

   $= \frac{d}{dx}\cos(3x + x^{1/2}) = \boxed{(3 + \frac{1}{2}x^{-\frac{1}{2}})(-\sin(3x + x^{1/2}))}$

3. $\frac{d}{dx}\sin(x)\cos(x) \quad = \sin(x)(-\sin(x)) + \cos(x)\cos(x)$

   $= \boxed{\cos^2(x) - \sin^2(x)}$

   Notice: $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, and $\cos^2(x) - \sin^2(x) = \cos(2x)$.

   Does your answer still make sense from this perspective?

4. $\frac{d}{dx}\sin(\cos(x^2 + 2)) = \cos(\cos(x^2 + 2)) * \frac{d}{dx}(\cos(x^2 + 2))$

   $= \cos(\cos(x^2 + 2)) * (-\sin(x^2 + 2)) * \frac{d}{dx}(x^2 + 2)$

   $= \boxed{\cos(\cos(x^2 + 2)) * (-\sin(x^2 + 2)) * (2x)}$

On your own, fill in the rest of the trig functions:

1. $\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)} = \boxed{\sec^2(x)}$

2. $\frac{d}{dx}\cot(x) = \frac{d}{dx}\frac{\cos(x)}{\sin(x)} = \boxed{-\csc^2(x)}$

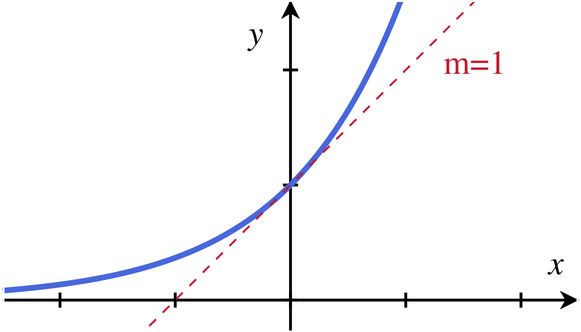
3. $\frac{d}{dx}\sec(x) = \frac{d}{dx}(\cos(x))^{-1} = \boxed{\sec(x)\tan(x)}$

4. $\frac{d}{dx}\csc(x) = \frac{d}{dx}(\sin(x))^{-1} = \boxed{-\csc(x)\cot(x)}$

The Derivative of $y = e^x$

Recall!

e^x is the unique exponential function whose slope at $x = 0$ is 1:



$$\lim_{h\to 0} \frac{e^{0+h} - e^0}{h} = \boxed{\lim_{h\to 0} \frac{e^h - 1}{h} = 1}$$

The Derivative of $y = e^x$...

$$\boxed{\lim_{h \to 0} \frac{e^h - 1}{h} = 1}$$

$$\begin{aligned}
\frac{d}{dx} e^x &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x \left(e^h - 1\right)}{h} \\
&= e^x \lim_{h \to 0} \frac{e^h - 1}{h} \\
&= e^x * 1
\end{aligned}$$

So

$$\boxed{\frac{d}{dx} e^x = e^x}$$

Examples

Calculate...

1. $\frac{d}{dx}e^{17x} = 17e^{17x}$

2. $\frac{d}{dx}e^{x\ln(3)} = \ln(3)e^{x\ln(3)}$

3. $\frac{d}{dx}e^{\sin x} = \cos(x)e^{\sin x}$

4. $\frac{d}{dx}e^{\sqrt{x^2+x}} = \frac{2x+1}{2\sqrt{x^2+x}}e^{\sqrt{x^2+x}}$

Notice, every time:

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

Other bases

We know $\frac{d}{dx}x^a = ax^{a-1}$, but what is $\frac{d}{dx}a^x$?
Notice:

$$a^x = e^{\ln(a^x)} = e^{x ln(a)}$$

So

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln(a)} = \ln(a)e^{x\ln(a)} = \ln(a) * a^x$$

Example: $\frac{d}{dx}2^x = \ln(2) * 2^x$

Sanity check: $\frac{d}{dx}e^x \stackrel{?}{=} \ln(e) * e^x = 1 * e^x$ ☺

Derivative so far

Definition: $f'(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Combining functions:

$$\frac{d}{dx} c * f(x) = c * f'(x) \qquad \frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

$$\frac{d}{dx}(f(x) * g(x)) = f(x)g'(x) + g(x)f'(x) \qquad \frac{d}{dx}(f(g(x)) = f'(g(x)) * g'(x)$$

Basic functions:

$f(x)$	x^a	$\sin(x)$	$\cos(x)$	e^x
$f'(x)$	ax^{a-1}	$\cos(x)$	$-\sin(x)$	e^x

Other functions:

$f(x)$	$\tan(x)$	$\cot(x)$	$\sec(x)$	$\csc(x)$	a^x
$f'(x)$	$\sec^2(x)$	$-\csc^2(x)$	$\sec(x)\tan(x)$	$-\csc(x)\cot(x)$	$\ln(a)a^x$

Example

Compute the derivatives of

1. $e^{17x^2+\sqrt{x}} = e^{17x^2+\sqrt{x}} * \left(17 * 2x + \frac{1}{2\sqrt{x}}\right)$
2. $\tan\left(e^{17x^2+\sqrt{x}}\right)$
   $$= \sec^2\left(e^{17x^2+\sqrt{x}}\right) * \left(e^{17x^2+\sqrt{x}} * \left(17 * 2x + \frac{1}{2\sqrt{x}}\right)\right)$$
3. $\csc(x) * \left[3^x + \tan^3\left(e^{17x^2+\sqrt{x}}\right)\right]^4$

Computing $\frac{d}{dx}\left[\csc(x) * \left[3^x + \tan^3\left(e^{17x^2+\sqrt{x}}\right)\right]^4\right]$:

first, the "shape" of this function is

$$a(x) * b\Big(c(x) + d\Big(f\Big(g\big(K \cdot h(x) + j(x)\big)\Big)\Big)\Big)$$

where

$a(x) = \csc(x)$

$b(x) = x^4$

$c(x) = 3^x$

$d(x) = x^3$

$f(x) = \tan(x)$

$g(x) = e^x$

$K = 17$

$h(x) = x^2$

$j(x) = \sqrt{x}$

Calculate:

$a'(x) = -\csc(x)\cot(x)$

$b'(x) = 4x^3$

$c'(x) = \ln(3) * 3^x$

$d'(x) = 3x^2$

$f'(x) = \sec^2(x)$

$g'(x) = e^x$

$h'(x) = 2x$

$j'(x) = \frac{1}{2\sqrt{x}}$

$$\frac{d}{dx}\csc(x) * \left(3^x + \tan^3\left(e^{17x^2+\sqrt{x}}\right)\right)^4$$

$$= \boxed{-\csc(x)\cot(x)} * \left(3^x + \tan^3\left(e^{17x^2+\sqrt{x}}\right)\right)^4 + \boxed{\csc(x)} * \boxed{\star}$$

$a'(x)$ $a(x)$

$$\frac{d}{dx}\left(3^x + \tan^3\left(e^{17x^2+\sqrt{x}}\right)\right)^4 = \boxed{4\left(3^x + \tan^3\left(e^{17x^2+\sqrt{x}}\right)\right)^3} * \boxed{\star}$$

$b'\left(3^x + \tan^3\left(e^{17x^2+\sqrt{x}}\right)\right)$

$$\frac{d}{dx}\left(3^x + \tan^3\left(e^{17x^2+\sqrt{x}}\right)\right) = \boxed{\ln(3)\, 3^x} + \boxed{3\left(\tan\left(e^{17x^2+\sqrt{x}}\right)\right)^2} * \boxed{\star}$$

$c'(x)$ $d'\left(\tan\left(e^{17x^2+\sqrt{x}}\right)\right)$

$$\frac{d}{dx}\tan\left(e^{17x^2+\sqrt{x}}\right) = \boxed{\sec^2\left(e^{17x^2+\sqrt{x}}\right)} * \boxed{\star}$$

$f'\left(e^{17x^2+\sqrt{x}}\right)$

$$\frac{d}{dx} e^{17x^2+\sqrt{x}} = \boxed{e^{17x^2+\sqrt{x}}} * \boxed{17*2x + \frac{1}{2\sqrt{x}}}$$

$g'\left(17x^2+\sqrt{x}\right)$ $\frac{d}{dx}\left(17x^2+\sqrt{x}\right)$

Put it all back together:

$$\frac{d}{dx} \csc(x) * \left[3^x + \tan^3\left(e^{17x^2+\sqrt{x}}\right)\right]^4$$

$$= -\csc(x)\cot(x) * \left(3^x + \tan^3\left(e^{17x^2+\sqrt{x}}\right)\right)^4$$

$$+ \csc(x) * 4\left[3^x + \tan^3\left(e^{17x^2+\sqrt{x}}\right)\right]^3$$

$$* \left[\ln(3)\, 3^x + 3\left(\tan\left(e^{17x^2+\sqrt{x}}\right)\right)^2 * \sec^2\left(e^{17x^2+\sqrt{x}}\right) * e^{17x^2+\sqrt{x}} * \left(17*2x + \frac{1}{2\sqrt{x}}\right)\right]$$