Differentiation Rules

Warm up

Use the limit definition of the derivative to calculate the following derivatives.

1. $\frac{d}{dx}(5x+2)$ $= 5$
2. $\frac{d}{dx}(3x-1)$ $= 3$
3. $\frac{d}{dx}\left[(5x+2)+(3x-1)\right]$ $= 8$
4. $\frac{d}{dx}\left[(5x+2)(3x-1)\right]$ $= 30x+1$
5. $\frac{d}{dx}15x^2$ $= 30x$
6. $\frac{d}{dx}(15x^2+x-2)$ $= 30x+1$

Remember the power rule says $\frac{d}{dx}x^a = ax^{a-1}$.
Based on your calculations above, which of the following statements seem to be true and which seem to be false?

(a) If you multiply a function $f(x)$ by a number c and then take a derivative, you get the same thing as taking the derivative $f'(x)$ and then multiplying by c. (try comparing 5 to the power rule) true?

(b) If you add two functions $f(x)$ and $g(x)$ and take a derivative, you get the same answer as taking the derivatives $f'(x)$ and $g'(x)$ and then adding those together. (try comparing 1-3, and then 6 to the power rule) true?

(c) If you multiply two functions $f(x)$ and $g(x)$ and take a derivative, you get the same answer as taking the derivatives $f'(x)$ and $g'(x)$ and then multiplying those together. (try comparing 1, 2, and 4) false!

1. $\frac{d}{dx}(5x+2) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{5(x+h)+2-(5x+2)}{h}$

$= \lim_{h\to 0} \frac{5\cancel{h}}{\cancel{h}} = \boxed{5}$

2. $\frac{d}{dx}(3x-1) = \lim_{h\to 0} \frac{(3(x+h)-1)-(3x-1)}{h} = \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}} = \boxed{3}$

3. $\frac{d}{dx}\left[(5x+2)+(3x-1)\right] = \frac{d}{dx}\left[8x+1\right] = \lim_{h\to 0} \frac{8(x+h)+1-(8x+1)}{h}$

simplify first

$= \lim_{h\to 0} \frac{8\cancel{h}}{\cancel{h}} = \boxed{8}$

4. $\frac{d}{dx}\left[(5x+2)(3x-1)\right] = \frac{d}{dx}\left(15x^2+x-2\right)$

$= \lim_{h\to 0} \frac{15(x+h)^2+(x+h)-2-(15x^2+x-2)}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left(15x^2+\underline{\underline{30xh}}+\underline{\underline{15h^2}}+x+\underline{\underline{h}}-2-15x^2-x+2\right)$

$= \lim_{h\to 0} 30x+\underline{15h}+1 = \boxed{30x+1}$

5. $\frac{d}{dx}[15x^2] = \lim_{h\to 0} \frac{15(x+h)^2 - 15x^2}{h}$

$= \lim_{h\to 0} \frac{15x^2 + 30xh + 15h^2 - 15x^2}{h}$

$= \lim_{h\to 0} 30x + \underbrace{15h}_{\to 0} = \boxed{30x}$

6. Notice $(5x+2)(3x-1) = 15x^2 + x - 2$,

so by #4, $\frac{d}{dx}(15x^2 + x - 2) = 30x + 1$.

Power rule: $\frac{d}{dx} 1 = 0$, $\frac{d}{dx} x = 1$, $\frac{d}{dx} x^2 = 2x$.

(a) Look at 5: We calculated $\frac{d}{dx} 15x^2 = 30x$, which $\underline{\underline{\text{is}}}$ $15 * \frac{d}{dx} x^2$.

(b) In 1, we showed $\frac{d}{dx} 5x+2 = \boxed{5}$

In 2, we showed $\frac{d}{dx} 3x-1 = \boxed{3}$

In 3, we got

$\frac{d}{dx}(5x+2)+(3x-1) = 8$, which $\underline{\underline{\text{is}}}$ $5+3$.

(c) In 1, we got $\frac{d}{dx} 5x+2 = 5$
and in 2 we got $\frac{d}{dx} 3x-1 = 3$,

but in 4, we got

$$\frac{d}{dx}(5x+2)(3x-1) = 30x+1$$

which is <u>not</u> $5 * 3 = 15$.

Multiplying by constants: what's going on?

Take another look at $f(x) = 15x^2$. Before, we just expanded and canceled, and were surprised to find something nice happened:

$$\frac{d}{dx}\left[15x^2\right] = \lim_{h\to 0} \frac{15(x+h)^2 - 15x^2}{h}$$

$$= \lim_{h\to 0} \frac{15x^2 + 30xh + 15h^2 - 15x^2}{h}$$

$$= \lim_{h\to 0} 30x + \underbrace{15h}_{\to 0} = \boxed{30x}$$

Let's try again, only pay closer attention to that 15:

$$\frac{d}{dx}15x^2 = \lim_{h\to 0} \frac{15(x+h)^2 - 15x^2}{h} = \lim_{h\to 0} 15 * \left(\frac{(x+h)^2 - x^2}{h}\right)$$

one of our limit rules!

$$= 15 * \underbrace{\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}}_{= \frac{d}{dx}x^2}$$

Let's try again, only pay closer attention to that 15:

$$\frac{d}{dx}15x^2 = \lim_{h\to 0} \frac{15(x+h)^2 - 15x^2}{h} = \lim_{h\to 0} 15 * \left(\frac{(x+h)^2 - x^2}{h}\right)$$

$$\overset{\text{one of our limit rules!}}{=} 15 * \underbrace{\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}}_{= \frac{d}{dx} x^2}$$

But now suppose you have any differentiable function $f(x)$ and a number c. [Think: $f(x) = x^2$ and $c = 15$]. Then *in general*

$$\frac{d}{dx}\left(c * f(x)\right) = \lim_{h\to 0} \frac{c * f(x+h) - c * f(x)}{h}$$

$$= \lim_{h\to 0} c * \left(\frac{f(x+h) - f(x)}{h}\right)$$

$$\overset{\text{limit rule}}{\overset{*}{=}} c * \lim_{h\to 0} \frac{(f(x+h) - f(x))}{h} = c * \frac{d}{dx} f(x)$$

Multiplying by constants

Theorem (Scalars)

If $y = f(x)$ is a differentiable function and c is a constant, then

$$\frac{d}{dx}(c * f(x)) = c * \frac{d}{dx} f(x).$$

Example

Since $\frac{d}{dx} x^2 = 2x$, we have $\frac{d}{dx} 15x^2 = 15 * 2x = 30x$.

Taking sums: what's going on?

Take another look at $f(x) = (5x+2) + (3x-1)$. Before, we just simplified first, and were surprised:

$$\frac{d}{dx}\left[(5x+2)+(3x-1)\right] \underset{\text{simplify first}}{=} \frac{d}{dx}\left[8x+1\right] = \lim_{h\to 0} \frac{8(x+h)+1-(8x+1)}{h}$$

$$= \lim_{h\to 0} \frac{8\cancel{h}}{\cancel{h}} = \boxed{8}$$

Let's try again, only pay closer attention to either part of the sum:

go together! go together!

$$\frac{d}{dx}\left((5x+2)+(3x-1)\right) = \lim_{h\to 0} \frac{(5(x+h)+2)+(3(x+h)-1)-[5x+2+3x-1]}{h}$$

$$= \lim_{h\to 0} \frac{[(5(x+h)+2)-(5x+2)]+[(3(x+h)-1)-(3x-1)]}{h}$$

because $\frac{a+b}{c} = \frac{a}{c}+\frac{b}{c}$

$$= \lim_{h\to 0} \frac{[5(x+h)+2]-(5x+2)}{h} + \frac{[3(x+h)-1]-(3x-1)}{h}$$

limit rule!

$$\overset{*}{=} \lim_{h\to 0} \frac{5(x+h)+2-(5x+2)}{h} + \lim_{h\to 0} \frac{3(x+h)-1-(3x-1)}{h}$$

$$= \frac{d}{dx}(5x+2) + \frac{d}{dx}(3x-1)$$

Now, suppose you have any differentiable functions $f(x)$ and $g(x)$ [Think: $f(x) = 5x + 2$ and $g(x) = 3x - 1$]. Then *in general*

go together! go together!

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \lim_{h\to 0} \frac{(f(x+h)+g(x+h)) - (f(x)+g(x))}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right]$$

(because $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$)

limit rule! *

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Theorem (Sums)

If f and g are differentiable functions, then

$$\frac{d}{dx}(f(x)+g(x)) = f'(x) + g'(x)$$

Example

Use the three rules we have so far

$$\frac{d}{dx}x^a = ax^{a-1}, \quad \frac{d}{dx}c * f(x) = c * \left(\frac{d}{dx}f(x)\right),$$

$$\text{and} \quad \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

to calculate the derivatives:

1. $\frac{d}{dx}(x^3 - 7x^2 + 6x^{-15})$

   $= \frac{d}{dx}x^3 - 7 * \frac{d}{dx}x^2 + 6 * \frac{d}{dx}x^{-15} = \boxed{3x^2 - 7 * 2x + 6(-15)x^{-16}}$

2. $\frac{d}{dx}\left(\sqrt{x} + 100\sqrt[17]{x^3} - \frac{3}{x^{19}}\right) = \frac{d}{dx}\left(x^{1/2} + 100x^{3/17} - 3x^{-19}\right)$

   $= \frac{d}{dx}x^{1/2} + 100 * \frac{d}{dx}x^{3/17} - 3 * \frac{d}{dx}x^{-19}$

   $= \boxed{\frac{1}{2}x^{-1/2} + 100 * \frac{3}{17}x^{-14/17} - 3 * (-19)x^{-20}}$

[hint: rewrite everything from 2 as powers before you do anything]

Products: What's going on?

Take another look at $f(x) = (5x+2)*(3x-1)$. Before, we just simplified first, and were. . . not surprised:

$$\frac{d}{dx}\left[(5x+2)(3x-1)\right] = \frac{d}{dx}\left(15x^2 + x - 2\right)$$

$$= \lim_{h\to 0} \frac{15(x+h)^2 + (x+h) - 2 - (15x^2 + x - 2)}{h}$$

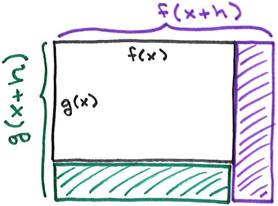
$$= \lim_{h\to 0} \frac{1}{h}\left(15x^2 + \underline{\underline{30xh}} + \underline{\underline{15h^2}} + x + \underline{\underline{h}} - 2 - 15x^2 - x + 2\right)$$

$$= \lim_{h\to 0} 30x + \underline{15h} + 1 = \boxed{30x+1}$$

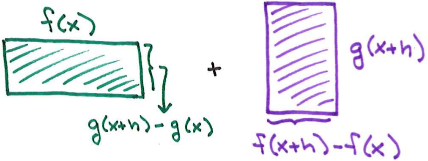
We *didn't* get that the derivative of the products is the product of the derivatives! So what *is* going on here?

To understand how to deal with products, we're going to have to unpack the formula

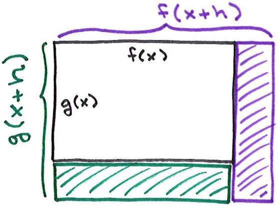
$$\frac{d}{dx} f(x) * g(x) = \lim_{h \to 0} \frac{f(x+h) * g(x+h) - f(x) * g(x)}{h}$$



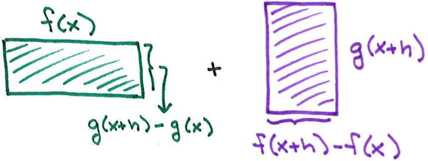
$f(x+h) * g(x+h) - f(x) * g(x) =$



$$= f(x) * \big(g(x+h) - g(x)\big) + g(x+h) * \big(f(x+h) - f(x)\big)$$



$$f(x+h) * g(x+h) - f(x) * g(x) =$$



$$= f(x) * \big(g(x+h) - g(x)\big) + g(x+h) * \big(f(x+h) - f(x)\big)$$

So

$$\frac{d}{dx} f(x) * g(x) = \lim_{h \to 0} \frac{f(x+h) * g(x+h) - f(x) g(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \Big(f(x) * [g(x+h) - g(x)] + g(x+h)[f(x+h) - f(x)] \Big)$$

So

$$\frac{d}{dx} f(x)*g(x) = \lim_{h\to 0} \frac{f(x+h)*g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(f(x)*[g(x+h)-g(x)] + g(x+h)[f(x+h)-f(x)]\right)$$

$$= \lim_{h\to 0}\left[f(x)*\left(\frac{g(x+h)-g(x)}{h}\right) + g(x+h)*\left(\frac{f(x+h)-f(x)}{h}\right)\right]$$

$$= f(x)*\boxed{\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}} \to g'(x)$$

$$+ \left(\underbrace{\lim_{h\to 0} g(x+h)}_{g(x)}\right)*\left(\underbrace{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}_{f'(x)}\right)$$

$$= f(x)*g'(x) + g(x)*f'(x)$$

Theorem (Products)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\cdot g'(x) + g(x)\cdot f'(x).$$

Example: Calculate $\frac{d}{dx}\big((5x+2)(3x-1)\big)$:

$$\frac{d}{dx}\big((5x+2)(3x-1)\big) = (5x+2)\cdot 3 + (3x-1)\cdot 5 = \boxed{\text{30x+1}} \ ☺$$

$$f\uparrow \qquad g\uparrow \qquad\qquad f \ \cdot\ g' \ +\ g \ \cdot\ f'$$

Last rule: Compositions.

Example: Calculate $\frac{d}{dx}(5x+2)^{100}$.
If $f(x) = x^{100}$ and $g(x) = 5x+2$, then $f(g(x)) = (5x+2)^{100}$.
So since $f'(x) = 100x^{99}$ and $g'(x) = 5$, if everything were right and just in the world, we would hope that

$$\frac{d}{dx}(5x+2)^{100} = 100(5)^{99}$$

But it's not!!

$$\boxed{\frac{d}{dx}(5x+2)^{100} \neq 100(5)^{99}}$$

Theorem (Chain rule)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}((f\circ g)(x)) = \frac{d}{dx}\big(f(g(x))\big) = f'(g(x))\cdot g'(x).$$

Last rule: Compositions.

Theorem (Chain rule)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}((f \circ g)(x)) = \frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

We won't prove this identity, but we can kind of see where it's coming from:

$$\frac{d}{dx} f(g(x)) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}}_{\text{how } f \text{ changes versus } g(x) \text{ (instead of versus } x) \; = f'(g(x))} * \underbrace{\frac{g(x+h) - g(x)}{h}}_{g'(x)}$$

Last rule: Compositions.

We won't prove this identity, but we can kind of see where it's coming from:

$$\frac{d}{dx} f(g(x)) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}}_{\substack{\text{how } f \text{ changes versus } g(x) \\ \text{(instead of versus } x\text{)} \\ = f'(g(x))}} * \underbrace{\frac{g(x+h) - g(x)}{h}}_{g'(x)}$$

In Leibniz notation:

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg} * \frac{dg}{dx}$$

$$\text{Chain rule:} \quad \frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Example

Calculate $\frac{d}{dx}(5x+2)^{100}$.

Here,

$$f(x) = x^{100} \quad \text{and} \quad g(x) = 5x+2.$$

So

$$f'(x) = 100x^{99} \quad \text{and} \quad g'(x) = 5$$

and so

$$\frac{d}{dx}(5x+2)^{100} = 100(5x+2)^{99} \cdot 5.$$

Chain rule: $\dfrac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$

Example

Calculate $\frac{d}{dx}\left(\sqrt{x^7+5}\right)$.

Here,

$$f(x) = \sqrt{x} = x^{1/2} \quad \text{and} \quad g(x) = x^7 + 5.$$

So

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad g'(x) = 7x^6$$

and so

$$\frac{d}{dx}\big(\sqrt{x^7+5}\big) = \frac{1}{2\sqrt{x^7+5}} \cdot 7x^6.$$

Derivative rules

In summary, the derivative rules we have now are

1. **Power rule:** $\frac{d}{dx}x^a = ax^{a-1}$
2. **Scalar rule:** $\frac{d}{dx}c * f(x) = c * \frac{d}{dx}f(x)$
3. **Sum rule:** $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
4. **Product rule:** $\frac{d}{dx}(f(x) * g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$
5. **Chain rule:** $\frac{d}{dx}f(g(x)) = f'(g(x)) * g'(x)$

Examples

Use everything you know to calculate the derivatives of

1. $(3x^2 + x + 1)(5x + 1)$
2. $(3x^2 + x + 1)(5x + 1)^2$
3. $(5x + 1)^{10}$
4. $(3x^2 + x + 1)(5x + 1)^{10}$
5. $\dfrac{\sqrt{x^2 - x}}{x + x^{-1}}$
6. $\dfrac{1}{\sqrt[3]{x^2 + 7x^{1/2}}}$

Use the derivative rules (not limits) to prove the identities

a. **Reciprocal identity**: $\dfrac{d}{dx}\dfrac{1}{f(x)} = -\dfrac{f'(x)}{f^2(x)}$

b. **Quotient identity**: $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

c. **Many products identity**:

$$\frac{d}{dx}(f(x) * g(x) * h(x) * k(x)) = \Big(f(x)g(x)h(x)\Big) * k'(x) + \Big(f(x)g(x)k(x)\Big) * h'(x) + \Big(f(x)h(x)k(x)\Big) * g'(x) + \Big(g(x)h(x)k(x)\Big) * f'(x)$$

a. Reciprocal identity:

Rewrite $\frac{1}{f(x)} = (f(x))^{-1}$

So, using chain rule, $\frac{d}{dx}\frac{1}{f(x)} = \frac{d}{dx}(f(x))^{-1}$

$$= -(f(x))^{-2} * f'(x)$$

(chain rule)

$$= -\frac{f'(x)}{f^2(x)}$$ ✓

b. Quotient rule

Rewrite $f(x)/g(x) = f(x) * (g(x))^{-1}$

So

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{d}{dx} f(x) * (g(x))^{-1}$$

$$= f'(x) * (g(x))^{-1} + f(x) * \left(-(g(x))^{-2}\right) * g'(x)$$

(product and then chain rule)

$$= \frac{f'(x)}{g(x)} * \frac{g(x)}{g(x)} - \frac{f(x)g'(x)}{g^2(x)}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$ ✓

c. Many Products identity:

$$\frac{d}{dx}\left(f(x) * \left(g(x) * h(x) * k(x)\right)\right) = f'(x) * g(x)h(x)k(x) + f(x) * \frac{d}{dx}\left(g(x)h(x)k(x)\right)$$

$$\frac{d}{dx}\left(g(x)\left(h(x) * k(x)\right)\right) = g'(x) * \left(h(x)k(x)\right) + g(x) * \frac{d}{dx}\left(h(x)k(x)\right)$$

$$\frac{d}{dx}\left(h(x) * k(x)\right) = h'(x)k(x) + k'(x)h(x).$$

put it back together...

Calculate $\frac{d}{dx}$ of ...

① $(3x^2+x+1)(5x+1)$

Two ways:

ⓐ expand first.

$$(3x^2+x+1)(5x+1) = 15x^3+8x^2+6x+1$$

$$\Big\downarrow \frac{d}{dx}$$

$$45x^2+16x+6$$

ⓑ product rule:

$$\frac{d}{dx}(3x^2+x+1)(5x+1)$$

$$= (3x^2+x+1)\cdot 5 + (5x+1)(6x+1)$$

"done" here.
but to compare to ⓐ ...

$$= 15x^2+5x+5 + 30x^2+11x+1$$

$$= 45x^2+16x+6$$

:)

② $y = (3x^2+x+1)(5x+1)^2$

Lots of ways, including:

ⓐ Expand all the way:

$$(3x^2+x+1)(5x+1)^2 = 75x^4 + 55x^3 + 38x^2 + 11x + 1$$

so $$\frac{dy}{dx} = 4\cdot 75x^3 + 3\cdot 55x^2 + 2\cdot 38x + 11$$
$$= \boxed{300x^3 + 165x^2 + 76x + 11}$$

ⓑ Expand the $(5x+1)^2$:

$$(3x^2+x+1)(5x+1)^2 = (3x^2+x+1)(25x^2+10x+1)$$

$\frac{d}{dx}$

product rule:

$$(3x^2+x+1)(50x+10) + (25x^2+10x+1)(6x+1)$$

"done"

$$= 150x^3 + 80x^2 + 60x + 10$$
$$+ 150x^3 + 85x^2 + 16x + 1$$
$$= 300x^3 + 165x^2 + 76x + 1$$

ⓒ two product rules:

$$\frac{d}{dx}\Big(\big((3x^2+x+1)(5x+1)\big)(5x+1)\Big)$$

$$= \underbrace{(3x^2+x+1)(5x+1)\cdot 5} + (5x+1)\cdot\underbrace{\frac{d}{dx}(3x^2+x+1)(5x+1)}$$

$$= 75x^3 + 40x^2 + 30x + 5 + (5x+1)\left[(3x^2+x+1)\cdot 5 + (5x+1)(6x+1)\right]$$

$$= \cdots = 300x^3 + 165x^2 + 76x + 1.$$

(d) My favorite:

product and chain rule:

$$\frac{d}{dx}\,(3x^2 + x + 1)(5x+1)^2$$

$$\left.\begin{aligned} &= (3x^2+x+1)\cdot 2(5x+1)^1\cdot 5 \\ &\quad + (5x+1)^2\cdot(6x+1) \end{aligned}\right\}\ \text{done.}$$

$$\begin{aligned} &= 150x^3 + 80x^2 + 60x + 10 \\ &\quad + 150x^3 + 85x^2 + 16x + 1 \\ &= 300x^3 + 165x^2 + 76x + 11 \end{aligned}$$

③ $y = (5x+1)^{10}$

Lots of ways, for example...

ⓐ Be obnoxious and expand first:

$$(5x+1)^{10} = 9{,}765{,}625\,x^{10} + 19{,}531{,}250\,x^9 + 17{,}578{,}125\,x^8 + 9{,}375{,}000\,x^7 + 3{,}281{,}250\,x^6 + 787{,}500\,x^5 + 131{,}250\,x^4 + 15{,}000\,x^3 + 1125\,x^2 + 50x + 1$$

so

$$\frac{dy}{dx} = 97{,}656{,}250\,x^9 + 175{,}781{,}250\,x^8 + 140{,}625{,}000\,x^7 + 65{,}625{,}000\,x^6 + 19{,}687{,}500\,x^5 + 3{,}937{,}500\,x^4 + 525{,}000\,x^3 + 45{,}000\,x^2 + 2250x + 50.$$

ⓑ Chain rule:

$$\frac{d}{dx}(5x+1)^{10} = 10(5x+1)^9 * 5$$

(which happens to be if you expand)

④ $(3x^2+x+1)(5x+1)^{10}$

Product, then chain:

$$(3x^2+x+1)\cdot\frac{d}{dx}(5x+1)^{10} + (5x+1)^{10}\frac{d}{dx}(3x^2+x+1)$$

$$= (3x^2+x+1)\cdot 10(5x+1)^9\cdot 5$$
$$+ (5x+1)^{10}(6x+1).$$

↑ "done",

but notice there's a quick rout to factorization:

pull out $(5x+1)^9$ that the two terms have in common:

$$= (5x+1)^9\left(50(3x^2+x+1) + (5x+1)(6x+1)\right)$$

$$= (5x+1)^9\left(150x^2+50x+50+30x^2+11x+1\right)$$

$$= (5x+1)^9\left(\underbrace{180x^2+66x+51}_{\text{no real roots!}}\right)$$

5 $\dfrac{\sqrt{x^2-x}}{x+x^{-1}}$

Many ways, including...

(a) Quotient rule: $y = \dfrac{f}{g}$

where

$$f = (x^2-x)^{1/2} \quad \text{\&} \quad g = x + x^{-1}$$

so

$$f' = \tfrac{1}{2}(x^2-x)^{-1/2} \quad \text{\&} \quad g' = 1 - x^{-2}$$

(2x−1) *

so $$\frac{dy}{dx} = \frac{f'g - g'f}{g^2} = \frac{\tfrac{1}{2}(x^2-x)^{-1/2}\cdot(x+x^{-1}) - (1-x^{-2})(x^2-x)^{1/2}}{(x+x^{-1})^2}$$

"done"

(b) product rule: $f\cdot(g)^{-1}$

where $f = (x^2-x)^{1/2}$ & $g = x + x^{-1}$

$$\frac{d}{dx} f\cdot(g)^{-1} = f\cdot\frac{d}{dx}(g)^{-1} + (g)^{-1}\frac{d}{dx}f$$

$$= (x^2-x)^{1/2}\cdot\left(-(x+x^{-1})^{-2}\cdot(1-x^{-2})\right)$$

$$+ (x+x^{-1})^{-1}\cdot\tfrac{1}{2}(x^2-x)^{-1/2}$$

← "done"

$$= -\frac{(x^2-x)^{1/2}(1-x^2)}{(x+x^{-1})^2} + \frac{(x+x^{-1})\cdot\tfrac{1}{2}(x^2-x)^{-1/2}}{(x+x^{-1})^2}$$

← same as (a) ✓

⑥ $\dfrac{1}{\sqrt[3]{x^2+7x^{1/2}}}$

Many ways, including...

ⓐ reciprocal rule: $\dfrac{1}{f}$

where $f = (x^2 + 7x^{\frac{1}{2}})^{1/3}$

since $f' = \frac{1}{3}(x^2 + 7x^{1/2})^{-2/3} \cdot (2x + 7/2\, x^{-1/2})$

$$\frac{dy}{dx} = -\frac{\frac{1}{3}(x^2+7x^{1/2})^{-2/3}(2x + \frac{7}{2}x^{-1/2})}{(\sqrt[3]{x^2+7x^{1/2}})^2}$$

↖ "done"

$$= -\frac{2x + \frac{7}{2}x^{-1/2}}{3(x^2+7x^{1/2})^{4/3}}.$$

← nicer.

ⓑ Rewrite first, then chain rule:

$$y = (x^2 + 7x^{1/2})^{-1/3}$$

so $\dfrac{dy}{dx} = -\frac{1}{3}(x^2+7x^{1/2})^{-4/3}(2x + \frac{7}{2}x^{-1/2})$

$$= -\frac{2x + 7/2\, x^{-1/2}}{3(x^2+7x^{1/2})^{4/3}},$$

just as before!