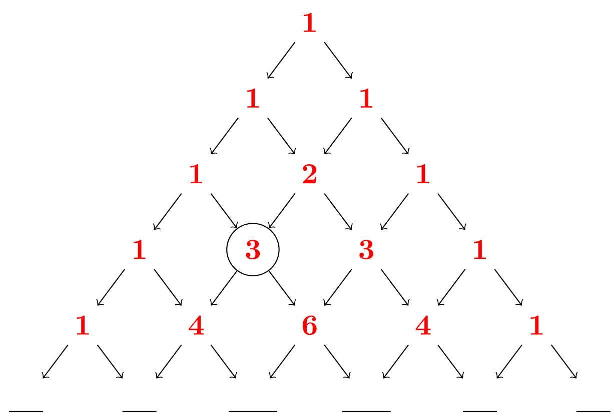
Warmup – quick expansions

Pascal's triangle:



To build Pascal's triangle:

- Start with 1's on the end.
- Add two numbers above to get a new entry.

For example, the circled 3 is the sum of the 1 and 2 above.

(A) Fill in the missing numbers.
Check: your answers should be symmetric (palindromic).

Pascal's triangle is useful for us because its makes expanding things like $(a+b)^n$ quickly–it gets its coefficients from the n^{th} row (where you start counting from 0). For example:

$$(a+b)^0 = \mathbf{1}$$
$$(a+b)^1 = \mathbf{1} * a + \mathbf{1} * b$$
$$(a+b)^2 = \mathbf{1} * a^2 + \mathbf{2} * ab + \mathbf{1} * b^2$$
$$(a+b)^3 = \mathbf{1} * a^3 + \mathbf{3} * a^2b + \mathbf{3} * ab^2 + \mathbf{1} * b^3$$
$$(a+b)^4 = \mathbf{1} * a^4 + \mathbf{4} * a^3b + \mathbf{6} * a^2b^2 + \mathbf{4} * ab^3 + \mathbf{1} * b^4$$
$$(a+b)^5 =$$

(B) Fill in the last expansion. Be organized–start with a^5, and lower the a exponent and raise the b exponent each by 1 for each subsequent term. Read the coefficients off the the 5th row you computed above.

Now we can use this to do other kinds of expansions quickly too. For example,

$$\begin{aligned}(h+2)^3 &= \mathbf{1} * h^3 + \mathbf{3} * h^2 * 2 + \mathbf{3} * h * 2^2 + \mathbf{1} * 2^3 \\ &= h^3 + 6h^2 + 12h + 8 \\ (h-3)^4 &= \mathbf{1} * h^4 + \mathbf{4} * h^3(-3) + \mathbf{6} * h^2(-3)^2 + \mathbf{4} * h(-3)^3 + \mathbf{1} * (-3)^4 \\ &= h^4 - 12h^3 + 54h^2 - 108h + 81 \\ (3h-7)^2 &= \mathbf{1} * (3h)^2 + \mathbf{2} * (3h) * (-7) + \mathbf{1} * (-7)^2 \\ &= 9h^2 - 42h + 49\end{aligned}$$

(C) Now you try:

$(h-1)^5 =$

$(h+2)^4 =$

$(2h-1)^3 =$

Derivatives

Definition

The *derivative* of a function f is a new function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We will say that a function f is *differentiable* over an interval (a, b) if the derivative function $f'(x)$ exists at every point in (a, b).

(see worksheet for when derivatives *don't* exist)

Q. How is this the same or different from what we were doing last time with tangent lines?

A. Last time, we were calculating derivatives at individual points, and getting *numbers* for answers. Today, we'll calculate the *derivative function*, and get out answers with variables in them (do all the points at once).

Example: let $f(x) = x^2$

Derivatives at a point: If I first ask "what is $f'(2)$?", I could calculate

$$f'(2) = \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} 4 + h = \boxed{4}.$$

But then, if I ask "what is $f'(3)$?" we have to do it all over again.

Today's goal: Write down a *function* $f'(x)$ which has all the derivatives-at-a-point collected together.

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If a is a number, (like 2 or 3) then

$$f'(a) = \underbrace{\lim_{h\to 0}}_{\text{gets rid of the } h\text{'s}} \underbrace{\frac{f(a+h)-f(a)}{h}}_{\text{a function of } a\text{'s and } h\text{'s}} = \text{number}$$

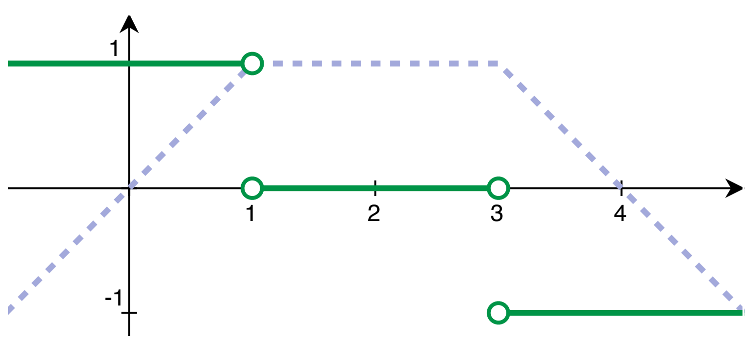
But x is a variable, so

$$f'(x) = \underbrace{\lim_{h\to 0}}_{\text{gets rid of the } h\text{'s}} \underbrace{\frac{f(x+h)-f(x)}{h}}_{\text{a function of } x\text{'s and } h\text{'s}} = \text{function of } x\text{'s}$$

Starting simple

Suppose we consider the piecewise linear function

$$f(x) = \begin{cases} x & x \le 1 \\ 1 & 1 < x < 3 \\ -x+4 & 3 \le x \end{cases}$$

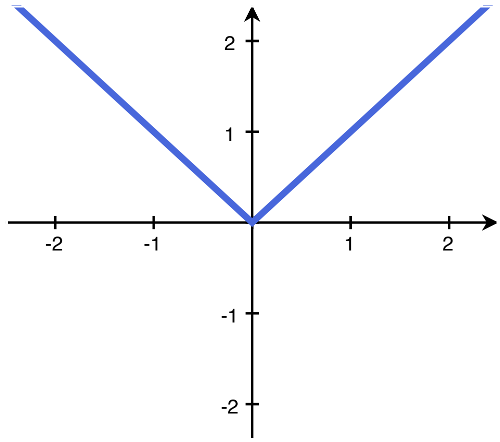


The derivative is:

$$f'(x) = \begin{cases} 1 & x < 1 \\ 0 & 1 < x < 3 \\ -1 & 3 < x \end{cases}$$

Another example

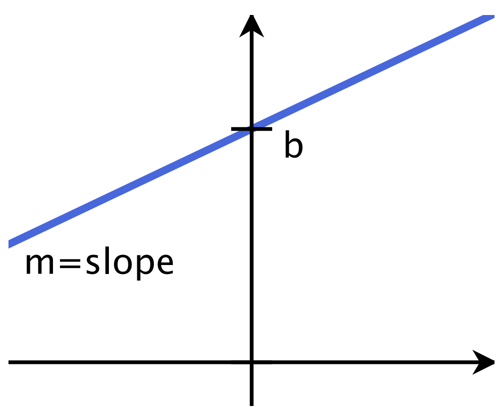
What is the derivative of $f(x) = |x|$?
Write down the piecewise function and sketch it on the graph.



Lines

In general, if m and b are constants, and

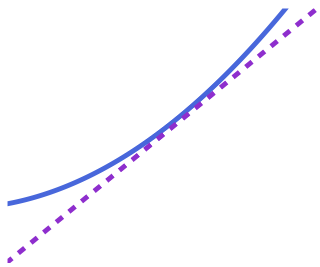
$$f(x) = mx + b$$



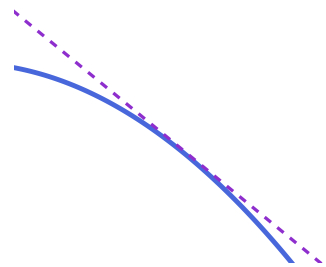
$$f'(x) = m$$

the slope of the tangent line = slope of the line

Rough shape of the derivative



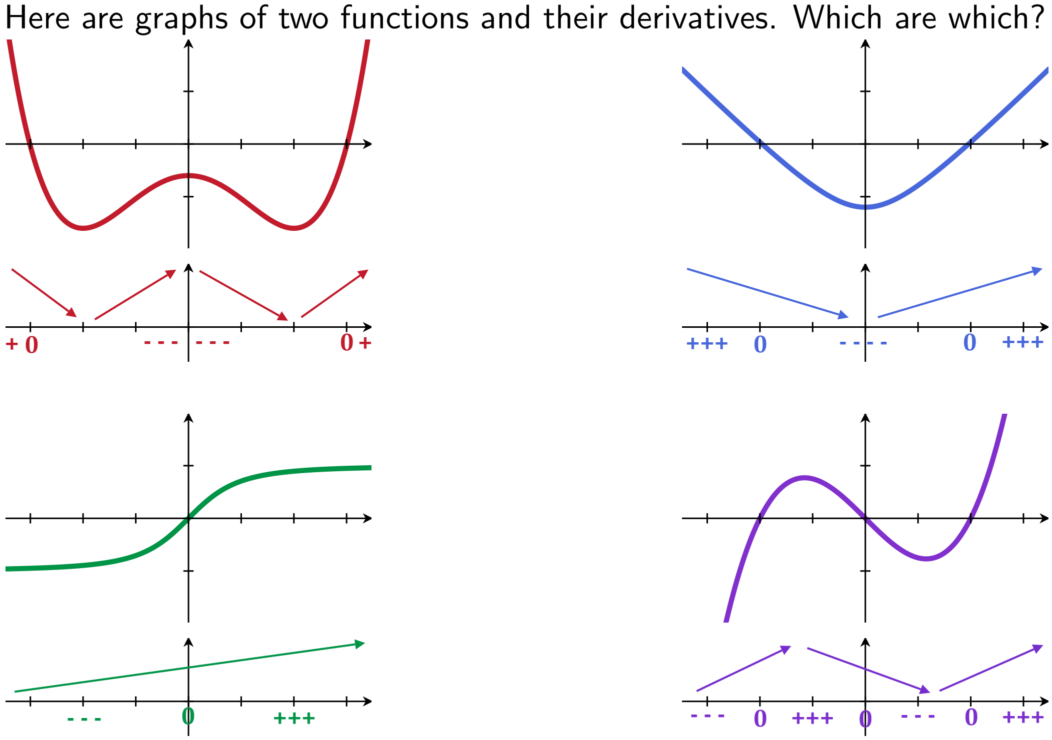
if $f(x)$ is increasing
$f'(x)$ is positive!



if $f(x)$ is decreasing
$f'(x)$ is negative!

Match em up!

Here are graphs of two functions and their derivatives. Which are which?

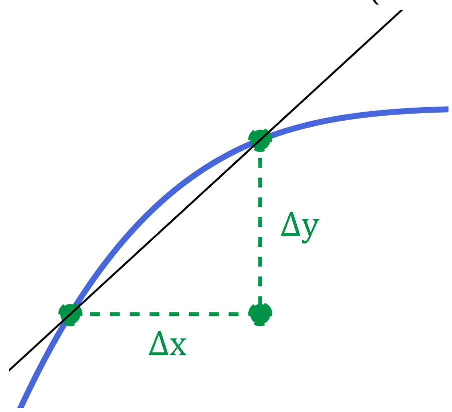


A little more notation

Back to what $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ means:

Rename $\boxed{h = \Delta x}$ and $\boxed{f(x+h) - f(x) = \Delta y}$

(Δ means "change")



So $m = \frac{f(x+h)-f(x)}{h} = \frac{\Delta y}{\Delta x}$.

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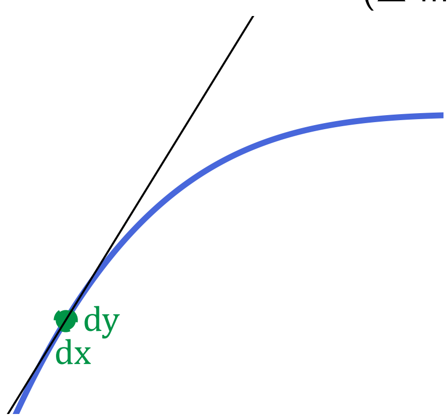

As $h \to 0$,
Δx and Δy get infinitely small.

$\Delta x \rightsquigarrow dx \qquad \Delta y \rightsquigarrow dy$

$$\frac{\Delta x}{\Delta y} \rightsquigarrow \frac{dy}{dx}$$

"infinitesimals"

So $m = \frac{f(x+h)-f(x)}{h} = \frac{\Delta y}{\Delta x}$.

Leibniz notation

One way to write the derivative of $f(x)$ versus x is $f'(x)$. Another way to write it is

$$f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x).$$

Derivatives at a point: $f'(a)$ means the derivative of $f(x)$ *evaluated at a*. Another way to write it is

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx} f(x)\right|_{x=a}$$

Example: We can write the derivative of x^2 as $\dfrac{d}{dx}x^2$

and the derivative of x^2 at $x = 5$ as $\left.\dfrac{d}{dx}x^2\right|_{x=5}$

Go to work: building our first derivative rule.

Example 1: What *is* the derivative of x^2?

$$\begin{aligned}
\frac{d}{dx}x^2 &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - (x)^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} 2x + h = \boxed{2x} \qquad \left(\text{so } \left.\frac{d}{dx}x^2\right|_{x=5} = 2 * 5\right)
\end{aligned}$$

By taking limits, fill in the rest of the table:

$f(x)$	1	x	x^2	x^3	$\frac{1}{x}$	$\frac{1}{x^2}$	$\sqrt{x}$	$\sqrt[3]{x}$
$f'(x)$			$2x$					

Hints: For $\frac{1}{x^2}$, find a common denominator, and then expand.

For $\sqrt[3]{x}$, try multiplying and dividing by $(\sqrt[3]{x+h})^2 + (\sqrt[3]{x+h})(\sqrt[3]{x}) + (\sqrt[3]{x})^2$.

$f(x)$	$f'(x)$
$x^0 = 1$	0
$x^1 = x$	1
x^2	$2x$
x^3	$3x^2$
$\frac{1}{x} = x^{-1}$	$-x^{-2}$
$\frac{1}{x^2} = x^{-2}$	$-2x^{-3}$
$\sqrt{x} = x^{1/2}$	$(1/2)x^{-1/2}$
$\sqrt[3]{x} = x^{1/3}$	$(1/3)x^{-2/3}$

Power rule:

$$\frac{d}{dx}x^a = ax^{a-1}$$

Use the power rule to take consecutive derivatives of $x^{5/2}$:

$$x^{5/2} \xrightarrow{\frac{d}{dx}} \boxed{\tfrac{5}{2}x^{3/2}} \quad 1^{st}\ \textit{derivative} = f'(x) = \frac{d}{dx}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{} \quad 2^{nd}\ \textit{derivative} = f''(x) = \frac{d^2}{dx^2}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{} \quad 3^{rd}\ \textit{derivative} = f^{(3)}(x) = \frac{d^3}{dx^3}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{}$$

$$4^{th}\ \textit{derivative} = f^{(4)}(x) = \frac{d^4}{dx^4}x^2$$

Definition: The n^{th} derivative of $f(x)$ is

$$\underbrace{\frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}}_{n} f(x) = \frac{d^n}{dx^n}f(x) = f^{(n)}(x).$$