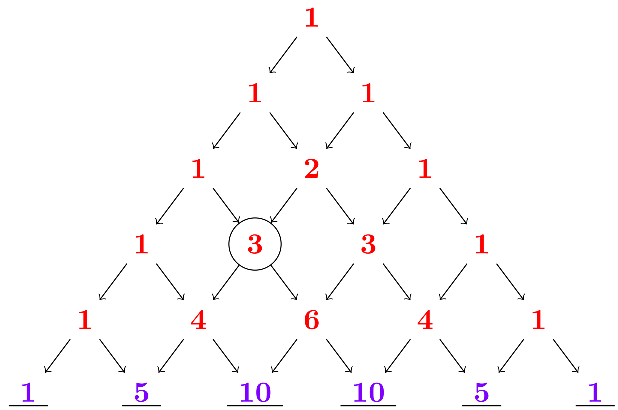
Warmup – quick expansions

Pascal's triangle:



To build Pascal's triangle:

- Start with 1's on the end.
- Add two numbers above to get a new entry.

For example, the circled (3) is the sum of the 1 and 2 above.

(A) Fill in the missing numbers.
Check: your answers should be symmetric (palindromic).

Pascal's triangle is useful for us because its makes expanding things like $(a+b)^n$ quickly–it gets its coefficients from the n^{th} row (where you start counting from 0). For example:

$$(a+b)^0 = \mathbf{1}$$
$$(a+b)^1 = \mathbf{1}*a + \mathbf{1}*b$$
$$(a+b)^2 = \mathbf{1}*a^2 + \mathbf{2}*ab + \mathbf{1}*b^2$$
$$(a+b)^3 = \mathbf{1}*a^3 + \mathbf{3}*a^2b + \mathbf{3}*ab^2 + \mathbf{1}*b^3$$
$$(a+b)^4 = \mathbf{1}*a^4 + \mathbf{4}*a^3b + \mathbf{6}*a^2b^2 + \mathbf{4}*ab^3 + \mathbf{1}*b^4$$
$$(a+b)^5 = \mathbf{1}*a^5 + \mathbf{5}*a^4b + \mathbf{10}*a^3b^2 + \mathbf{10}*a^2b^3 + \mathbf{5}*ab^4 + \mathbf{1}*b^5$$

(B) Fill in the last expansion. Be organized–start with a^5, and lower the a exponent and raise the b exponent each by 1 for each subsequent term. Read the coefficients off the the 5th row you computed above.

Now we can use this to do other kinds of expansions quickly too. For example,

$$\begin{aligned}(h+2)^3 &= \mathbf{1}*h^3 + \mathbf{3}*h^2*2 + \mathbf{3}*h*2^2 + \mathbf{1}*2^3\\ &= h^3 + 6h^2 + 12h + 8\\ (h-3)^4 &= \mathbf{1}*h^4 + \mathbf{4}*h^3(-3) + \mathbf{6}*h^2(-3)^2 + \mathbf{4}*h(-3)^3 + \mathbf{1}*(-3)^4\\ &= h^4 - 12h^3 + 54h^2 - 108h + 81\\ (3h-7)^2 &= \mathbf{1}*(3h)^2 + \mathbf{2}*(3h)*(-7) + \mathbf{1}*(-7)^2\\ &= 9h^2 - 42h + 49\end{aligned}$$

(C) Now you try:

$(h-1)^5 = \boxed{h^5 - 5h^4 + 10h^3 - 10h^2 + 5h - 1}$

$(h+2)^4 = h^4 + 4*2*h^3 + 6*2^2*h^2 + 4*2^3*h + 2^4 = \boxed{h^4 + 8h^3 + 24h^2 + 32h + 16}$

$(2h-1)^3 = (2h)^3 + 3*(2h)^2*(-1) + 3*(2h)*(-1)^2 + (-1)^3 = \boxed{8h^3 - 12h^2 + 6h - 1}$

Derivatives

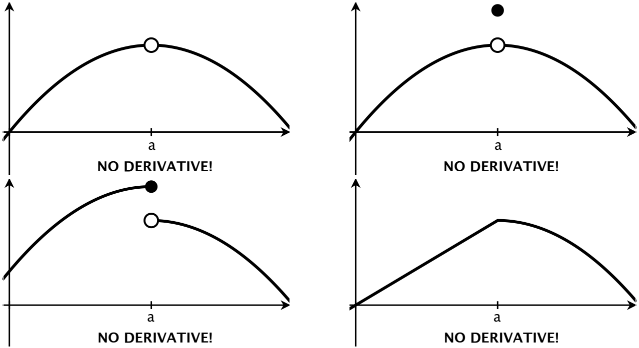
Definition

The *derivative* of a function f is a new function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We will say that a function f is *differentiable* over an interval (a, b) if the derivative function $f'(x)$ exists at every point in (a, b).

Remember from the worksheet:



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(see worksheet for when derivatives *don't* exist)

Q. How is this the same or different from what we were doing last time with tangent lines?

A. Last time, we were calculating derivatives at individual points, and getting *numbers* for answers. Today, we'll calculate the *derivative function*, and get out answers with variables in them (do all the points at once).

Example: let $f(x) = x^2$

Derivatives at a point: If I first ask "what is $f'(2)$?", I could calculate

$$f'(2) = \lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h\to 0} \frac{4h + h^2}{h} = \lim_{h\to 0} 4 + h = \boxed{4}.$$

But then, if I ask "what is $f'(3)$?" we have to do it all over again.

Today's goal: Write down a *function* $f'(x)$ which has all the derivatives-at-a-point collected together.

If a is a number, (like 2 or 3) then

$$f'(a) = \underbrace{\lim_{h\to 0}}_{\substack{\text{gets rid} \\ \text{of the } h\text{'s}}} \underbrace{\frac{f(a+h) - f(a)}{h}}_{\substack{\text{a function of} \\ a\text{'s and } h\text{'s}}} = \text{number}$$

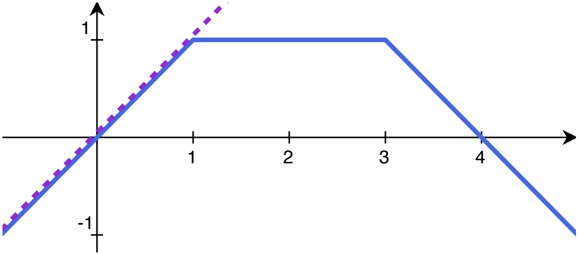
But x is a variable, so

$$f'(x) = \underbrace{\lim_{h\to 0}}_{\substack{\text{gets rid} \\ \text{of the } h\text{'s}}} \underbrace{\frac{f(x+h) - f(x)}{h}}_{\substack{\text{a function of} \\ x\text{'s and } h\text{'s}}} = \text{function of } x\text{'s}$$

Starting simple

Suppose we consider the piecewise linear function

$$f(x) = \begin{cases} x & x \leq 1 \\ 1 & 1 < x < 3 \\ -x + 4 & 3 \leq x \end{cases}$$



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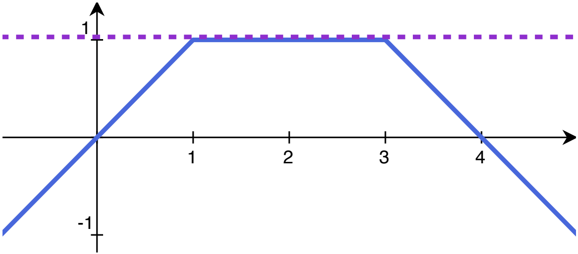

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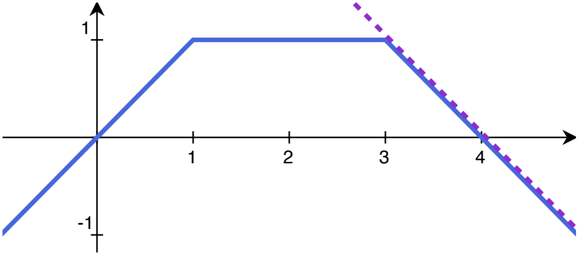

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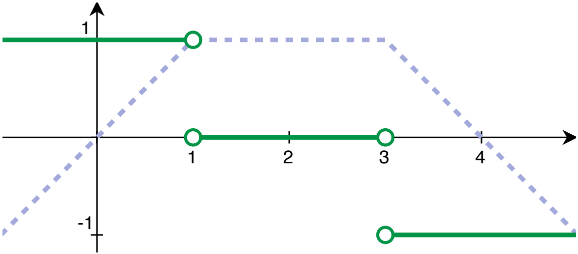


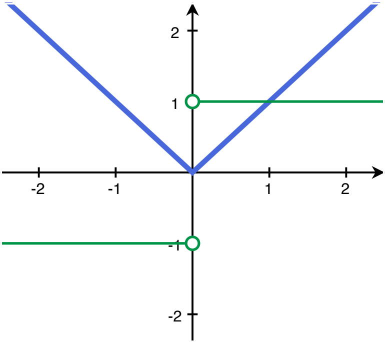
The derivative is:

$$f'(x) = \begin{cases} 1 & x < 1 \\ 0 & 1 < x < 3 \\ -1 & 3 < x \end{cases}$$

Another example

What is the derivative of $f(x) = |x|$?

Write down the piecewise function and sketch it on the graph.

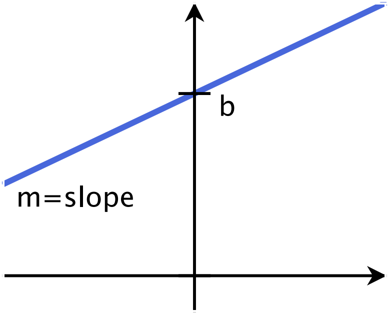


$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

Lines

In general, if m and b are constants, and

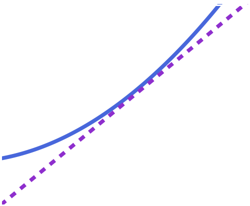
$$f(x) = mx + b$$



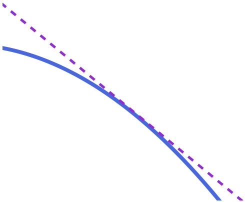
$$f'(x) = m$$

the slope of the tangent line = slope of the line

Rough shape of the derivative



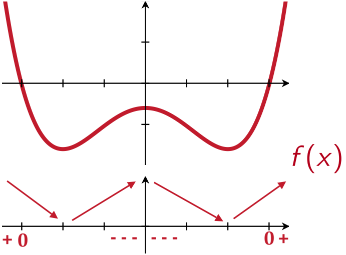
if $f(x)$ is increasing
$f'(x)$ is positive!

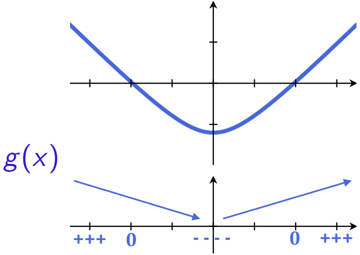


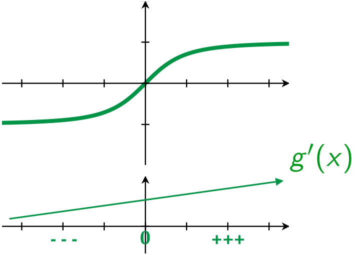
if $f(x)$ is decreasing
$f'(x)$ is negative!

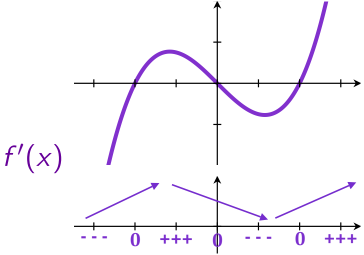
Match em up!

Here are graphs of two functions and their derivatives. Which are which?







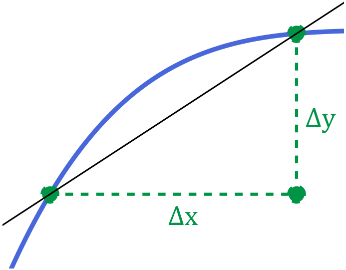


A little more notation

Back to what $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ means:

Rename $\boxed{h = \Delta x}$ and $\boxed{f(x+h) - f(x) = \Delta y}$

(Δ means "change")



So $m = \frac{f(x+h)-f(x)}{h} = \frac{\Delta y}{\Delta x}$.

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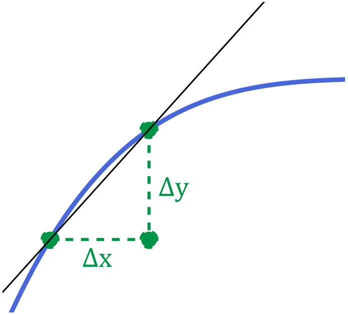

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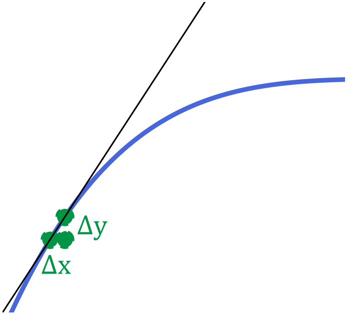

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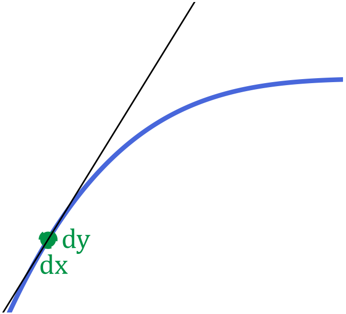

As $h \to 0$,
Δx and Δy get infinitely small.

$$\Delta x \rightsquigarrow dx \qquad \Delta y \rightsquigarrow dy$$

$$\frac{\Delta y}{\Delta x} \rightsquigarrow \frac{dy}{dx}$$

"infinitesimals"

So $m = \frac{f(x+h)-f(x)}{h} = \frac{\Delta y}{\Delta x}$.

Leibniz notation

One way to write the derivative of $f(x)$ versus x is $f'(x)$. Another way to write it is

$$f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x).$$

Derivatives at a point: $f'(a)$ means the derivative of $f(x)$ *evaluated at a*. Another way to write it is

$$f'(a) = \left. \frac{df}{dx} \right|_{x=a} = \left. \frac{d}{dx} f(x) \right|_{x=a}$$

Example: We can write the derivative of x^2 as $\frac{d}{dx} x^2$

and the derivative of x^2 at $x = 5$ as $\left. \frac{d}{dx} x^2 \right|_{x=5}$

Go to work: building our first derivative rule.

Example 1: What *is* the derivative of x^2?

$$\frac{d}{dx}x^2 = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x)^2}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h}$$

$$= \lim_{h\to 0} 2x+h = \boxed{2x} \qquad \left(\text{so } \frac{d}{dx}x^2\Big|_{x=5} = 2*5\right)$$

By taking limits, fill in the rest of the table:

$f(x)$	1	x	x^2	x^3	$\frac{1}{x}$	$\frac{1}{x^2}$	$\sqrt{x}$	$\sqrt[3]{x}$
$f'(x)$			$2x$					

Hints: For $\frac{1}{x^2}$, find a common denominator, and then expand.

For $\sqrt[3]{x}$, try multiplying and dividing by $(\sqrt[3]{x+h})^2 + (\sqrt[3]{x+h})(\sqrt[3]{x}) + (\sqrt[3]{x})^2$.

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$f'(x)$	0	1	$2x$	$3x^2$	$-\frac{1}{x^2}$	$-\frac{2}{x^3}$	$\frac{1}{2\sqrt{x}}$	$\frac{1}{3(\sqrt[3]{x})^2}$

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$f(x) = x$

$$\lim_{h \to 0} \frac{(x+h) - x}{h} \overset{*}{=} \lim_{h \to 0} \frac{h}{h} = \boxed{1}$$

$f(x) = x^2$

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$\overset{*}{=} \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h$$

$$= \boxed{2x}$$

$f(x) = x^3$

recall: $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

so $(x+h)^3 - x^3 = 3x^2h + 3xh^2 + h^3$.

So, if $h \neq 0$

$$* \quad \frac{(x+h)^3 - x^3}{h} = 3x^2 + 3xh + h^2$$

So

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \left(3x^2 + 3xh + h^3 \right)$$

$$= \boxed{3x^2} + 0 + 0$$

$$f(x) = \frac{1}{x} = x^{-1}$$

$$\lim_{h \to 0} \frac{\left(\frac{1}{x+h} - \frac{1}{x}\right)}{h} = \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{x-(x+h)}{(x+h)(x)}\right)$$

$$\overset{*}{=} \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{-h}{(x+h)x}\right)$$

$$= \lim_{h \to 0} -\frac{1}{(x+h)x} = -\frac{1}{x^2} = \boxed{-x^{-2}}$$

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$\lim_{h \to 0} \frac{\left(\frac{1}{(x+h)^2} - \frac{1}{x^2}\right)}{h} = \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{x^2-(x+h)^2}{(x+h)^2 x^2}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{x^2-(x^2+2xh+h^2)}{(x+h)^2 \; x^2}\right)$$

$$\overset{*}{=} \lim_{h \to 0} \frac{1}{h}\left(\frac{-2xh+h^2}{(x+h)^2 x^2}\right) = \lim_{h \to 0} \frac{-2x+h}{(x+h)^2 x^2}$$

$$= -\frac{2x}{x^4} = -\frac{2}{x^3} = \boxed{-2x^{-3}}$$

From last class:

$$f(x) = \sqrt{x}$$

so $f(x+h) = \sqrt{x+h}$

so

can't plug in $h=0$ yet

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \quad \begin{matrix}\to 0 \\ \to 0\end{matrix}$$

$$= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

so

$$\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2x^{1/2}}$$

$f(x) = \sqrt[3]{x} = x^{1/3}$:

Since

$$(a-b)(a^2+ab+b^2)$$

$$= a^3 + a^2b + ab^2$$

$$-(a^2b + ab^2 + b^3)$$

$$= a^3 - b^3,$$

let $a = \sqrt[3]{x+h}$
$b = \sqrt[3]{x}$

we have

$$\left((x+h)^{1/3} - x^{1/3}\right)\left((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}\right)$$

$$= (x+h)^{3/3} - x^{3/3} = x+h-x = h.$$

so

$$\lim_{h \to 0} \frac{(x+h)^{1/3} - x^{1/3}}{h} \cdot \frac{\left((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}\right)}{\left((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}\right)}$$

$$\overset{*}{=} \lim_{h \to 0} \frac{h}{h\left((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}\right)}$$

$$= \frac{1}{x^{2/3} + x^{1/3}x^{1/3} + x^{2/3}} = \frac{1}{3x^{2/3}} = \boxed{\frac{1}{3}x^{-2/3}}$$

$f(x)$	$f'(x)$
$x^0 = 1$	0
$x^1 = x$	1
x^2	$2x$
x^3	$3x^2$
$\frac{1}{x} = x^{-1}$	$-x^{-2}$
$\frac{1}{x^2} = x^{-2}$	$-2x^{-3}$
$\sqrt{x} = x^{1/2}$	$(1/2)x^{-1/2}$
$\sqrt[3]{x} = x^{1/3}$	$(1/3)x^{-2/3}$

Power rule:

$$\frac{d}{dx}x^a = ax^{a-1}$$

Use the power rule to take consecutive derivatives of $x^{5/2}$:

$$x^{5/2} \xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2}x^{3/2}} \quad \textit{1}^{st} \textit{ derivative} = f'(x) = \frac{d}{dx}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2}x^{1/2}} \quad \textit{2}^{nd} \textit{ derivative} = f''(x) = \frac{d^2}{dx^2}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2}x^{-1/2}} \quad \textit{3}^{rd} \textit{ derivative} = f^{(3)}(x) = \frac{d^3}{dx^3}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \left(-\frac{1}{2}\right) x^{-3/2}}$$

$$\textit{4}^{th} \textit{ derivative} = f^{(4)}(x) = \frac{d^4}{dx^4}x^2$$

Definition: The n^{th} derivative of $f(x)$ is

$$\underbrace{\frac{d}{dx}\frac{d}{dx}\dots\frac{d}{dx}}_{n} f(x) = \frac{d^n}{dx^n}f(x) = f^{(n)}(x).$$