

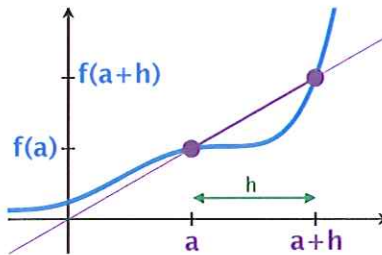
# Worksheet – The tangent line problem

We've been building towards studying rates of change, e.g.

- rate at which position changes versus time (= velocity);
- rate at which birthrate changes versus average household income;
- rate at which profit margin changes versus production volume.

In general, the instantaneous rate of change of a function  $f(x)$  versus  $x$  at a point  $a$  is given by the limit of the *difference quotient*:

$$\text{inst. rate of change} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$



Another word for the instantaneous rate of change of a function  $f(x)$  at a point  $a$  is the **derivative** of  $f(x)$  at  $x = a$ , written  $f'(a)$ . So

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

The derivative also has a *geometric* interpretation:

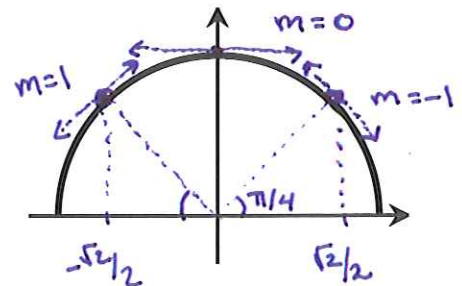
$$f'(a) = \text{slope of the line tangent to } y = f(x) \text{ at } x = a.$$

**Example 1:** Below is a graph of the function  $f(x) = \sqrt{1-x^2}$  (the half circle with radius 1). Without calculating any limits, what is

- (a)  $f'(0)$ ? = 0
- (b)  $f'(\frac{\sqrt{2}}{2})$ ? = -1
- (c)  $f'(-\frac{\sqrt{2}}{2})$ ? = 1

[hint: for (b) and (c), draw a line from the origin to the point in question. What angle does that make with the  $x$ -axis? What is the slope of that line? For a circle, the line tangent at a point is perpendicular to the ray from the center to the point.]

note:  $f(\frac{\sqrt{2}}{2}) = f(-\frac{\sqrt{2}}{2}) = \sqrt{1 - 2/4} = \sqrt{2}/2$



Once you have the slope, it's pretty easy to write down the equations for the tangent line using point-slope form:

$$y = m(x - x_0) + y_0 \quad \text{becomes} \quad \boxed{y = f'(a)(x - a) + f(a)}$$

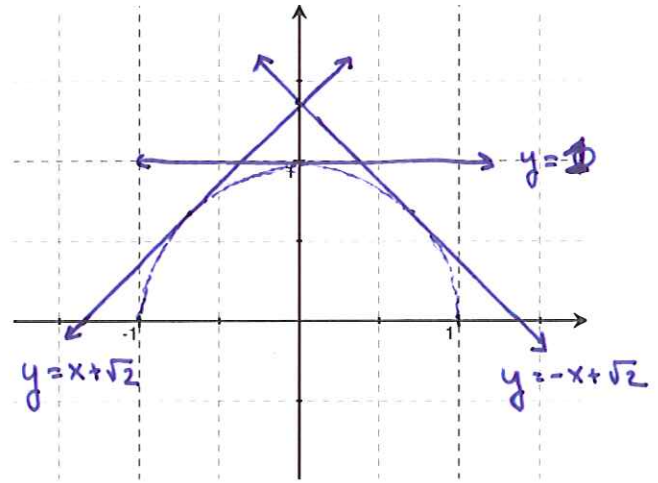
**Example 2:** What is the equation for the line tangent to  $f(x) = \sqrt{1 - x^2}$  at

(a)  $x = 0$ ?  $y = 0 \cdot (x - 0) + 1 = 1$

note:  $\sqrt{2} \approx 1.4$

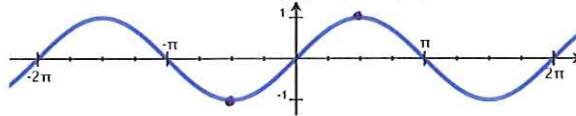
(b)  $x = \frac{\sqrt{2}}{2}$ ?  $y = -1 \cdot (x - \frac{\sqrt{2}}{2}) + \frac{\sqrt{2}}{2}$   
 $= -x + \sqrt{2}$

(c)  $x = -\frac{\sqrt{2}}{2}$ ?  $y = 1 \cdot (x - (-\frac{\sqrt{2}}{2})) + \frac{\sqrt{2}}{2}$   
 $= x + \sqrt{2}$



Check your answers by first sketching the lines you wrote down in (a)-(c), and then sketching the function  $f(x) = \sqrt{1 - x^2}$  on the axes to the right.

**Example 3:** For reference, the graph of  $f(x) = \sin(x)$  is:



(a) The function  $\sin(x)$  has infinitely many points  $x = a$  where  $f'(a) = 0$ . What are they?

$$x = \pi/2 + 2\pi k \quad \text{and} \quad x = -\pi/2 + 2\pi k$$

for  $k = 0, \pm 1, \pm 2, \dots$                       for  $k = 0, \pm 1, \pm 2, \dots$

(b) There are exactly two horizontal lines which are tangent to  $\sin(x)$ . What are they?

$$y = 1 \quad \text{and} \quad y = -1$$

(c) [Bonus] Can you think of a function which has infinitely many points where  $f'(a) = 0$ , not just anywhere, but between  $x = 0$  and  $x = \pi$ ? [hint: think back to the day we did limits. There is some function  $g(x)$  which we could plug into  $\sin(x)$  which will make  $\sin(g(x))$  a good answer to this question.]

$$y = \sin(1/x) \quad \text{or} \quad \cos(1/x) \quad \text{or} \quad \sec(1/x) \quad \text{etc.}$$

Answers: 1(a) : 0, (b) : -1, (c) : 1, 2(a) :  $y = 1$ , (b) :  $y = -x + \sqrt{2}$ , (c) :  $x + \sqrt{2}$ , 3(a) :  $\frac{\pi}{2} + \pi k$ , (b) :  $y = \pm 1$ .

## Calculating derivative using limits

Recall from last Friday that we have a few tricks for calculating limits  $\lim_{x \rightarrow a} g(x)$ :

1. **Plugging in:** If  $g(x)$  is continuous, and  $g(a)$  is defined, then  $\lim_{x \rightarrow a} g(x) = g(a)$ .

For example,  $\lim_{x \rightarrow 2} \frac{x+1}{x-3} = \boxed{-3}$

since at  $x=2$ ,  $f(2) = \frac{2+1}{2-3} =$

2. **Factor and cancel:** If  $g(x)$  is rational, and  $g(a)$  is not defined, but  $a$  is a root of the numerator and denominator, then factor and cancel:

For example,

$$\lim_{x \rightarrow 2} \frac{x+1}{x-2} \text{ is undefined, (since } x-2 \rightarrow 0, \text{ but } x+1 \rightarrow 3)$$

but

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{1}{x+2} = \boxed{\frac{1}{4}}$$

3. **Expand and cancel:** It's like spring cleaning – make a mess, and then clean up!

For example, since  $(x+2)^3 = x^3 + 6x^2 + 12x + 8$ ,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x}{(x+2)^3 - 8} &= \lim_{x \rightarrow 0} \frac{x}{x^3 + 6x^2 + 12x + 8 - 8} = \boxed{\frac{1}{12}} \\ &= \lim_{x \rightarrow 0} \frac{x}{x(x^2 + 6x + 12)} = \lim_{x \rightarrow 0} \frac{1}{x^2 + 6x + 12} \end{aligned}$$

4. **Common denominators:** If you have a sum or difference of fractions, find a common denominator and see what happens.

For example,

$$\begin{aligned} \lim_{x \rightarrow 0} \left( \frac{1}{x} - \frac{1}{x(x+1)} \right) &= \lim_{x \rightarrow 0} \left( \frac{x+1-1}{x(x+1)} \right) \\ &= \lim_{x \rightarrow 0} \frac{x}{x(x+1)} = \lim_{x \rightarrow 0} \frac{1}{x+1} = \boxed{1} \end{aligned}$$

5. **Multiply top and bottom by the conjugate:** If you have a difference of square roots (like  $\sqrt{a} - \sqrt{b}$ ), you can multiply and divide by the *conjugate*,  $\sqrt{a} + \sqrt{b}$ . This is useful because

$$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = \boxed{a - b}$$

For example, try multiplying by  $\frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2}$ : (notice  $2 = \sqrt{4}$ )

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x}{\sqrt{x+4} - 2} &= \lim_{x \rightarrow 0} \frac{x(\sqrt{x+4} + 2)}{(\sqrt{x+4} - 2)(\sqrt{x+4} + 2)} \\ &= \lim_{x \rightarrow 0} \frac{x(\sqrt{x+4} + 2)}{x+4 - 4} \\ &= \lim_{x \rightarrow 0} \frac{x(\sqrt{x+4} + 2)}{x} = \lim_{x \rightarrow 0} (\sqrt{x+4} + 2) = \boxed{4} \end{aligned}$$

Now that we have these tools, let's calculate some derivatives!

(A) Use the limit definitions the derivative of  $f(x) = x^2$  at  $x = 1$ :

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h)^2 - (1)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{1+2h+h^2-1}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2+h)}{h} \\ &= \lim_{h \rightarrow 0} 2+h \end{aligned} \rightarrow = \boxed{2}$$

(B) Use the limit definitions the derivative of  $f(x) = x^3$  at  $x = -2$ :

$$\begin{aligned} f'(-2) &= \lim_{h \rightarrow 0} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(-2+h)^3 - (-2)^3}{h} \quad \text{careful! } (-2)^3 = -8, \text{ so } -(-2)^3 = 8 \\ &= \lim_{h \rightarrow 0} \frac{-8+12h-6h^2+h^3+8}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(12-6h+h^2)}{h} \\ &= \lim_{h \rightarrow 0} 12-6h+h^2 \end{aligned} \rightarrow = \boxed{12}$$

(C) Use the limit definitions the derivative of  $f(x) = \frac{1}{x}$  at  $x = 3$ :

$$\begin{aligned}
 f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{1}{3+h} - \frac{1}{3} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{3 - (3+h)}{3(3+h)} \right) \\
 &= \lim_{h \rightarrow 0} \frac{-h}{h \cdot 3(3+h)} = \lim_{h \rightarrow 0} \frac{-1}{3(3+h)} \\
 &= \boxed{-\frac{1}{9}}
 \end{aligned}$$

(D) Use the limit definitions the derivative of  $f(x) = \sqrt{x}$  at  $x = 5$ :

$$\begin{aligned}
 f'(3) &= \lim_{h \rightarrow 0} \frac{f(5+h) - f(5)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{5+h} - \sqrt{5}}{h} \cdot \frac{(\sqrt{5+h} + \sqrt{5})}{(\sqrt{5+h} + \sqrt{5})} \\
 &= \lim_{h \rightarrow 0} \frac{5+h-5}{h(\sqrt{5+h} + \sqrt{5})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{5+h} + \sqrt{5}} \\
 &= \frac{1}{\sqrt{5} + \sqrt{5}} \\
 &= \boxed{\frac{1}{2\sqrt{5}}}
 \end{aligned}$$

Back to tangent line equations:

Use your answers to A-D on the previous two pages to calculate the lines tangent to

(a)  $f(x) = x^2$  at  $x = 1$

$$y = 2(x-1) + (1)^2 = 2x - 1$$

(b)  $f(x) = x^3$  at  $x = -2$

$$y = 12(x+2) + (-2)^3 = 12x + 16$$

(c)  $f(x) = \frac{1}{x}$  at  $x = 3$

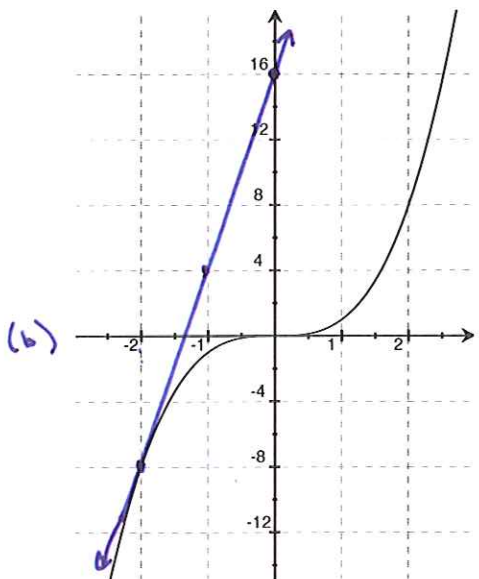
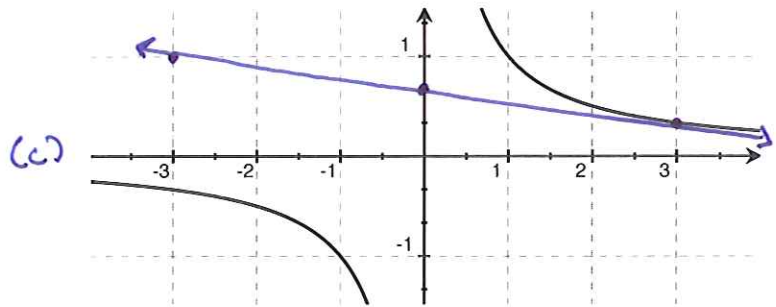
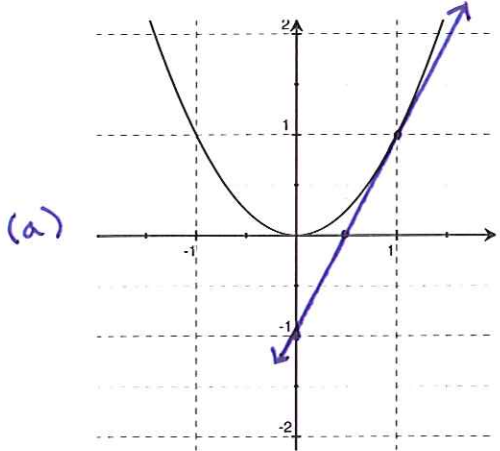
$$y = -\frac{1}{9}(x-3) + \frac{1}{3} = -\frac{1}{9}x + \frac{2}{3}$$

(d)  $f(x) = \sqrt{x}$  at  $x = 5$

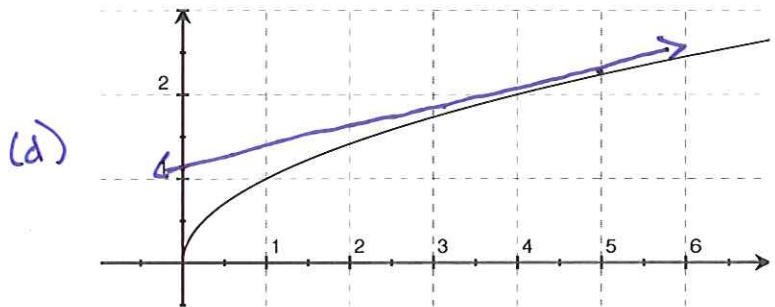
$$y = \frac{1}{2\sqrt{5}}(x-5) + \sqrt{5} = \frac{1}{2\sqrt{5}}x + \frac{\sqrt{5}}{2}$$

↘  $\frac{\sqrt{5}}{10}$

Check your answers by sketching the lines from (a)-(d) on onto the appropriate graphs below:



$$\frac{\sqrt{5}}{2} = 1.11\dots, \quad \frac{\sqrt{5}}{10} = .22\dots$$



Answers: a:  $y = 2x - 1$ , b:  $y = 12x + 16$ , c:  $-\frac{1}{9}x + \frac{2}{3}$ , d:  $\frac{1}{2\sqrt{5}}x + \frac{\sqrt{5}}{2}$

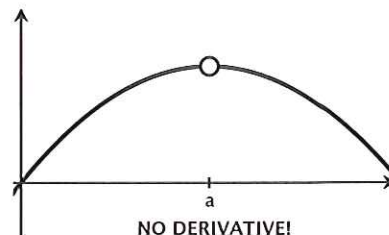
## When can we take derivatives?

Not all functions have derivatives at all places. Before calculating  $f'(a)$ , first ask ...

### 1. Is $f(x)$ defined at $x = a$ ?

For example, even if it looks like you could draw a tangent line, if there's a hole,  $f'(a)$  **does not exist!**

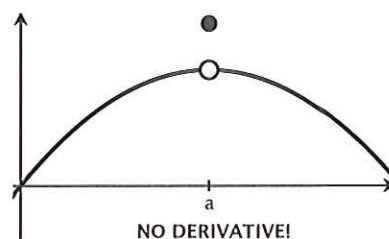
(It's tempting to say  $f'(a)$  exists here in part because  $f(x)$  has a *continuous extension* at  $a$ .)



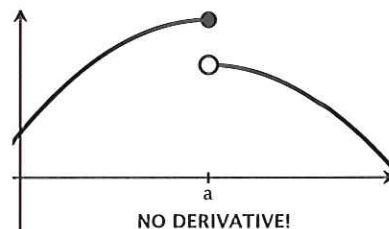
### 2. Is $f(x)$ continuous at $x = a$ ?

For example, even if it looks like you could draw a tangent line, if there's a jump,  $f'(a)$  **does not exist!**

(Try drawing just one line that is tangent to that isolated point. It's tempting to say  $f'(a)$  exists here in part because  $f(x)$  has a *removable discontinuity* at  $a$ .)



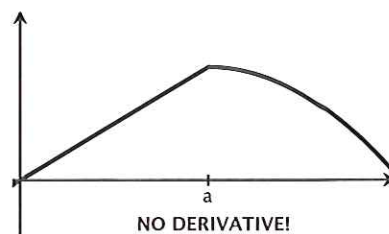
Again, even if the slope looks the same from the left and from the right, if there's a discontinuity,  $f'(a)$  **does not exist!**



### 3. Is there a "corner" at $x = a$ ?

Next we'll explore how to find these algebraically, but if there's a sharp corner at  $x = a$ , then  $f'(a)$  **does not exist!**

(Try drawing just one line that is tangent to that corner)





What's wrong with corners?

$$\text{Let } f(x) = \begin{cases} x^2 & x < 2, \\ x + 2 & x \geq 2. \end{cases}$$

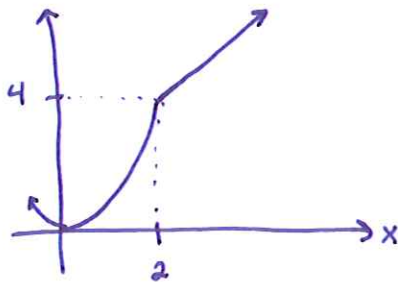
(a) Verify that  $f(x)$  is continuous at  $x = 2$ .

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} x^2 = \underline{4}.$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} x + 2 = \underline{4}.$$

$$f(2) = 2 + 2 = \underline{4}.$$

(b) Sketch a graph of  $f(x)$ .



(c) Estimate, and then calculate the *right sided derivative*.

(i) Estimate:

$a$	3	2.5	2.1	2	$h$	$f(2+h) - f(2)$	$\frac{f(2+h)-f(2)}{h}$
$f(a)$	5	4.5	4.1	4	1	1	1
					1/2	.5	1
					1/10	.1	1

(ii) Explain why  $\lim_{h \rightarrow 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0^-} \frac{(2+2+h) - (2+2)}{h}$ .

if  $h > 0$ , then  $f(2+h) = (2+h) + 2$ ,  
since  $2+h > 2$ .

(iii) Calculate  $\lim_{h \rightarrow 0^+} \frac{f(2+h) - f(2)}{h}$ .

$$= \lim_{h \rightarrow 0^+} \frac{2+2+h - (2+2)}{h}$$

$$= \lim_{h \rightarrow 0^+} h/h = 1.$$

(d) Estimate, and then calculate the left sided derivative. (OK to use a calculator for (i))

(i) Estimate:

$a$	1	1.5	1.9	2
$f(a)$	1	2.25	3.61	4

$h$	$f(2+h) - f(2)$	$\frac{f(2+h) - f(2)}{h}$
-1	-3	3
-1/2	-1.75	3.5
-1/10	-0.39	3.9

(ii) Explain why  $\lim_{h \rightarrow 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0^-} \frac{(2+h)^2 - (2)^2}{h}$ .

if  $h < 0$ , then  $2+h < 2$ ,  
so  $f(2+h) = (2+h)^2$ .

(iii) Calculate  $\lim_{h \rightarrow 0^-} \frac{f(2+h) - f(2)}{h}$ .

$$\begin{aligned}
 &= \lim_{h \rightarrow 0^-} \frac{(2+h)^2 - 4}{h} \\
 &= \lim_{h \rightarrow 0^-} \frac{4 + 4h + h^2 - 4}{h} \\
 &= \lim_{h \rightarrow 0^-} 4 + h = \boxed{4}
 \end{aligned}$$

(e) Compare your answers to (b) and (c), and explain why  $\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$  does not exist. Explain why  $f'(2)$  does not exist.

The left limit  
is 4

but

the right limit  
is 1

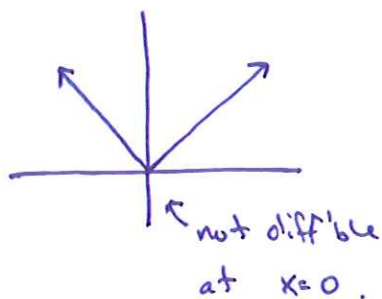
which are  
unequal,  
so the 2-sided  
limit does not  
exist.

(f) Sketch graphs of the following functions and identify points where each function is not differentiable:

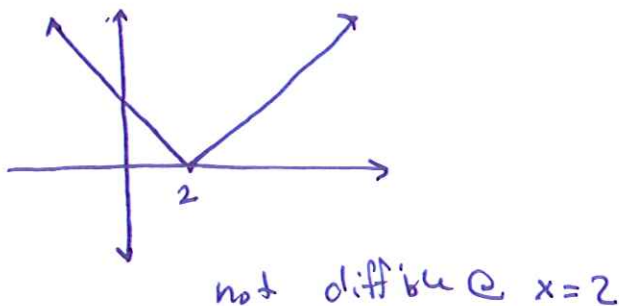
$$f(x) = |x| \quad g(x) = |x-2| \quad h(x) = |4-|x-2|| \quad \psi(x) = \frac{|x|}{x} \quad \phi(x) = \begin{cases} x^2 & x < 0, \\ x^4 & x \geq 0. \end{cases}$$

[hint: for  $h(x)$ , start by plotting some points, and then find points where  $x-2$  goes from positive to negative, and where  $4-|x-2|$  goes from positive to negative.]

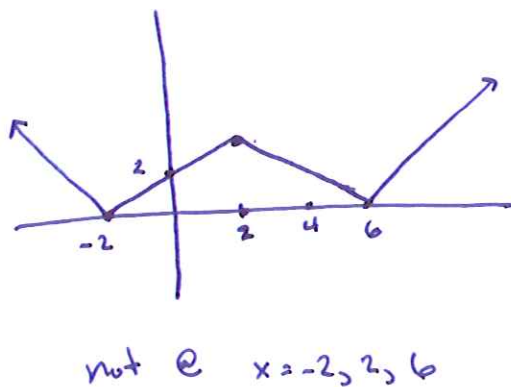
$$f(x) = |x|:$$



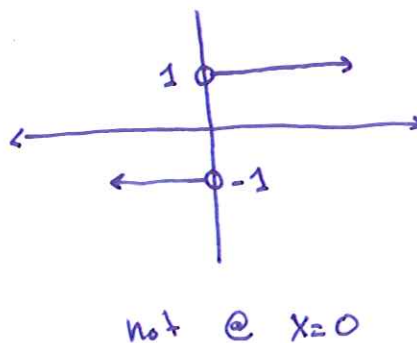
$$g(x) = |x-2|$$



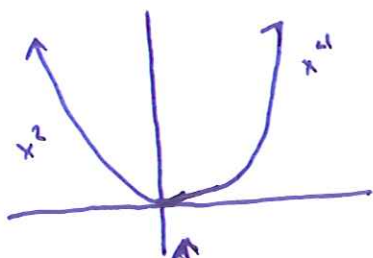
$$h(x) = |4-|x-2||$$



$$\psi(x) = \frac{|x|}{x}$$



\*even though the slope is 1 from both sides\*



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check!

$$\lim_{h \rightarrow 0^-} \frac{(0+h)^2 - 0^2}{h} = 0$$

$$\lim_{h \rightarrow 0^+} \frac{(0+h)^4 - 0^2}{h} = 0$$

ok! diff'ble @ all x.