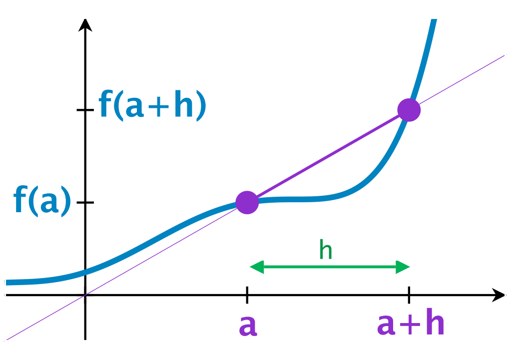
Limits

From last time...

Let $y = f(t)$ be a function that gives the position at time t of an object moving along the y-axis. Then

$$\begin{aligned}\text{Ave vel}[t_1, t_2] &= \frac{f(t_2) - f(t_1)}{t_2 - t_1} \\ \text{Velocity}(t) &= \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.\end{aligned}$$



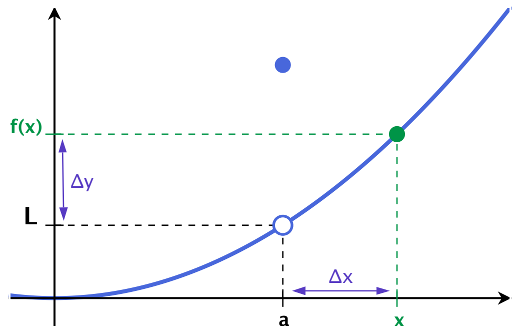
We need to be able to take limits!

Limit of a Function – Definition

We say that a function f *approaches the limit* L as x approaches a,

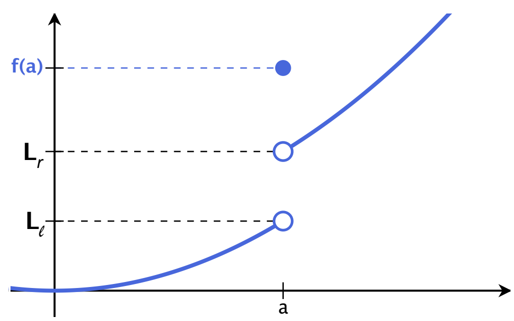
$$\text{written} \qquad \lim_{x \to a} f(x) = L,$$

if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a.



i.e. If you need Δy to be smaller,
you only need to make Δx smaller
(Δ means "change")

One-sided limits



Right-handed limit: $L_r = \lim_{x \to a^+} f(x)$

if $f(x)$ gets closer to L_r as x gets closer to a *from the right*

Left-handed limit: $L_\ell = \lim_{x \to a^-} f(x)$

if $f(x)$ gets closer to L_ℓ as x gets closer to a *from the left*

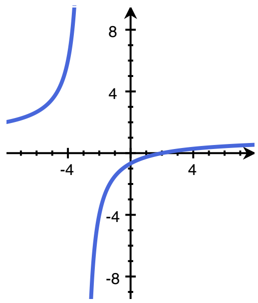
Theorem

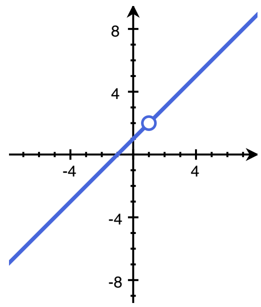
The limit of f as $x \to a$ exists if and only if both the right-hand and left-hand limits exist and have the same value, i.e.

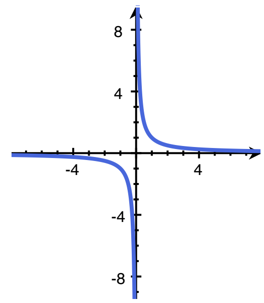
$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Examples

$$\lim_{x\to 2} \frac{x-2}{x+3} \qquad \lim_{x\to 1} \frac{x^2-1}{x-1} \qquad \lim_{x\to 0} \frac{1}{x}$$







Theorem

If $\lim_{x\to a} f(x) = A$ *and* $\lim_{x\to a} g(x) = B$ *both exist, then*

1. $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = A + B$
2. $\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = A - B$
3. $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A \cdot B$
4. $\lim_{x\to a}(f(x)/g(x)) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x) = A/B$ $(B \neq 0)$.

In short: to take a limit

Step 1: Can you just plug in? If so, do it.

Step 2: If not, is there some sort of algebraic manipulation (like cancellation) that can be done to fix the problem? If so, do it. Then plug in.

Step 3: Learn some special limit to fix common problems. (Later)

If in doubt, graph it!

Examples

1. $\lim_{x\to 2} \frac{x-2}{x+3} = \boxed{0}$ because if $f(x) = \frac{x-2}{x+3}$, then $f(2) = 0$.
2. $\lim_{x\to 1} \frac{x^2-1}{x-1} \begin{smallmatrix}\to 0\\ \to 0\end{smallmatrix}$

   If $f(x) = \frac{x^2-1}{x-1}$, then $f(x)$ is undefined at $x = 1$.
   However, so long as $x \neq 1$,

   $$f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1.$$

   So

   $$\lim_{x\to -1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 1+1 = \boxed{2}.$$

3. $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} \begin{smallmatrix}\to 0\\ \to 0\end{smallmatrix}$, so again, $f(x)$ is undefined at a.

Examples

3. $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} \begin{smallmatrix}\to 0\\ \to 0\end{smallmatrix}$, so again, $f(x)$ is undefined at a.

   Multiply top and bottom by the *conjugate*:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} &= \lim_{x\to 0} \left(\frac{\sqrt{x+2}-\sqrt{2}}{x}\right)\left(\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\right) \\
&= \lim_{x\to 0} \frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})} \qquad \text{since } (a-b)(a+b) = a^2-b^2 \\
&= \lim_{x\to 0} \frac{x}{x(\sqrt{x+2}+\sqrt{2})} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}
\end{aligned}$$

Examples

1. $\lim_{x \to 1} \frac{x^2-3x+2}{x^2+4x-5}$

2. $\lim_{x \to -2} \frac{|x|}{x}$

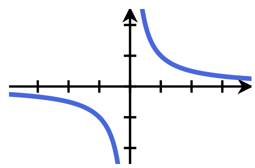
3. $\lim_{x \to 0} \frac{|x|}{x}$

4. $\displaystyle \lim_{x \to 0} \frac{(3+x)^2 - 3^2}{x}$

Infinite limits

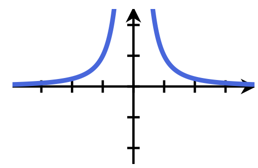
If $f(x)$ gets arbitrarily large as $x \to a$, then it doesn't have a limit. Sometimes, though, it's more useful to give more information.

Example:
For both $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$, $\lim_{x\to 0} f(x)$ does not exist. However, they're both "better behaved" than that might imply:



$$\lim_{x\to 0^+} \frac{1}{x} = \infty, \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist}$$



$$\lim_{x\to 0^+} \frac{1}{x^2} = \infty, \quad \lim_{x\to 0^-} \frac{1}{x^2} = \infty$$

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

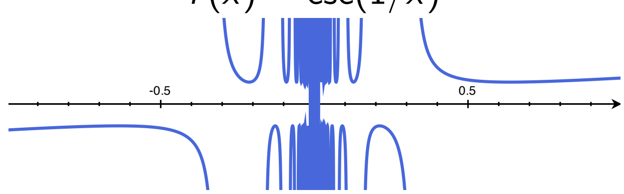
Why? A *vertical asymptote* occurs where

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

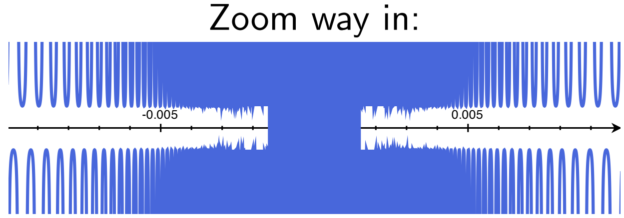
Infinite limits

Badly behaved example:

$$f(x) = \csc(1/x)$$



Zoom way in:



(denser and denser vertical asymptotes)

$$\lim_{x\to 0^+} \csc(1/x) \text{ does not exist, and } \lim_{x\to 0^-} \csc(1/x) \text{ does not exist}$$

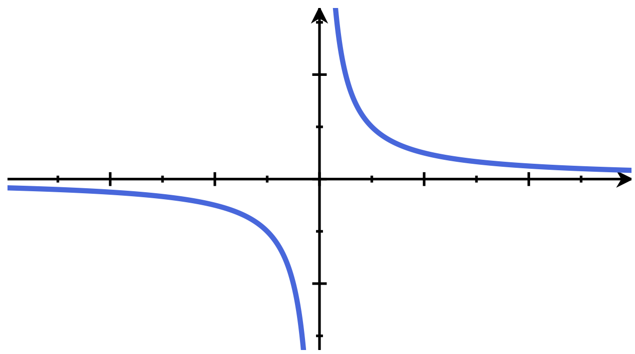
Limits at Infinity

We say that a function f *approaches the limit* L as x gets bigger and bigger (in the positive or negative direction),

$$\text{written} \qquad \lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

if we can make $f(x)$ as close to L as we want by taking x sufficiently large. *(By "large", we mean large in magnitude)*

Example 1:



$$\lim_{x\to-\infty} \frac{1}{x} = 0 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{1}{x} = 0.$$

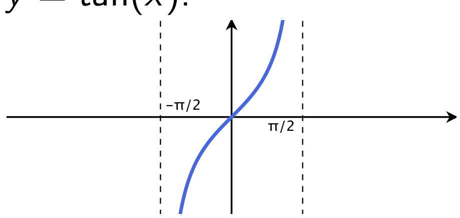
Limits at Infinity: functions and their inverses

Theorem

If $\lim_{x\to a^\pm} f(x) = \infty$, *then* $\lim_{x\to\infty} f^{-1}(x) = a$.
If $\lim_{x\to a^\pm} f(x) = -\infty$, *then* $\lim_{x\to-\infty} f^{-1}(x) = a$.

Example: Let $\arctan(x)$ be the inverse function to $\tan(x)$:

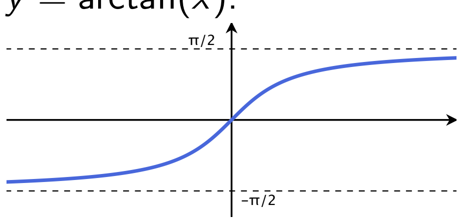
$y = \tan(x)$:



(restrict the domain to $(-\frac{\pi}{2}, \frac{\pi}{2})$ to that we can invert)

Since $\lim_{x\to\pi/2^-} = \infty$
and $\lim_{x\to-\pi/2^+} = -\infty$

$y = \arctan(x)$:



we have $\lim_{x\to\infty} = \pi/2$
and $\lim_{x\to-\infty} = -\pi/2$

Rational functions

Limits that look like they're going to $\frac{\infty}{\infty}$ can actually be doing lots of different things. To fix this, divide the top and bottom by the highest power in the denominator!

Ex 1 $\displaystyle \lim_{x\to\infty} \frac{x-1}{x^3+2} \begin{matrix}\to\infty \\ \to\infty\end{matrix}$

Fix: multiply the expression by $\frac{1/x^3}{1/x^3}$:

$$\lim_{x\to\infty} \frac{x-1}{x^3+2} = \lim_{x\to\infty} \left(\frac{x-1}{x^3+2}\right)\left(\frac{1/x^3}{1/x^3}\right) = \lim_{x\to\infty} \frac{\frac{1}{x^2}-\frac{1}{x^3}}{1+\frac{2}{x^3}} = \frac{0-0}{1+0} = \boxed{0}$$

Ex 2 $\displaystyle \lim_{x\to\infty} \frac{3x^2-2x+1}{4x^2-1} \begin{matrix}\to\infty \\ \to\infty\end{matrix}$

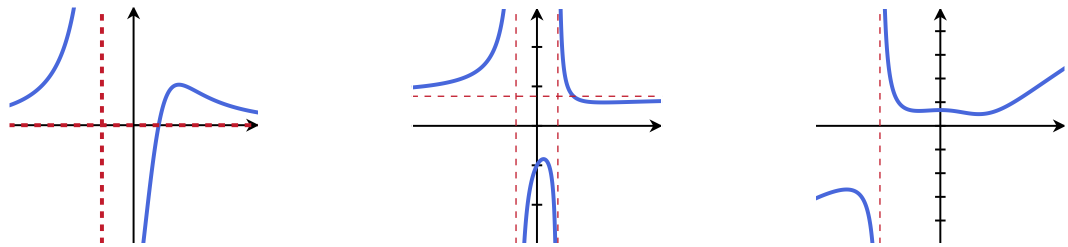
Fix: multiply the expression by $\frac{1/x^2}{1/x^2}$

Ex 3 $\displaystyle \lim_{x\to\infty} \frac{x^4-x^2+2}{x^3+3} \begin{matrix}\to\infty \\ \to\infty\end{matrix}$

Fix: multiply the expression by $\frac{1/x^3}{1/x^3}$

Rational functions: quick trick

$$\lim_{x\to\infty}\frac{x-1}{x^3+2}=0 \qquad \lim_{x\to\infty}\frac{3x^2-2x+1}{4x^2-1}=\frac{3}{4} \qquad \lim_{x\to\infty}\frac{x^4-x^2+2}{x^3+3}=\infty$$



Suppose

$$P(x)=a_nx^n+\cdots+a_1x+a_0 \quad \text{and} \quad Q(x)=b_mx^m+\cdots+b_1x+b_0$$

are polynomials of degree n and m respectively. Then in general,

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)}=\lim_{x\to\infty}\frac{a_nx^n}{b_mx^m}=\lim_{x\to\infty}\frac{a_n}{b_m}x^{n-m}=\begin{cases}0 & n<m\\ \frac{a_n}{b_m} & n=m\\ \pm\infty & n<m\end{cases}$$

Examples: Other ratios with powers.

Example:

$$\lim_{x\to\infty}\frac{x}{\sqrt{3x^2+2}}$$

[hint: multiply by $\frac{1/x}{1/x}$ and remember $a\sqrt{b}=\sqrt{a^2b}$.]

Evaluating limits when $x \to 0$.

1. Show $\lim_{x\to 0} (x^2-2)^2 + 6 = 10$.

2. Show $\lim_{x\to 0} \frac{5x}{x} = 5$.

3. Show $\lim_{x\to 0} \frac{17x}{2x} = 17/2$.

4. Show $\lim_{x\to 0} \frac{-317x}{422x} = -317/422$.

5. Show $\lim_{x\to 0} \frac{-317x-3}{422x+5} = -3/5$.

6. Show $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{1}{2\sqrt{x}}$.

7. Show $\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x} = 1/2$.

8. Show $\lim_{x\to 0} \frac{\sqrt{3+x}-\sqrt{3}}{x} = 1/(2\sqrt{3})$.

9. Show $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right) = -(1/2)x^{-3/2}$.

10. Show $\lim_{x\to 0} \frac{2x}{\sqrt{a+x}-\sqrt{a-x}} = 2\sqrt{a}$.

11. Show $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = 1/2$.

12. Show $\lim_{x\to 0} \frac{x}{\sqrt{1+x}-1} = 2$.

13. Show $\lim_{x\to 0} \frac{e^x+e^{-x}-2}{x^2} = 1$.

14. Show

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \frac{a}{2\sqrt{ax+b}}$$

when $f(x) = \sqrt{ax+b}$.

15. Show

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = mn(mx+c)^{n-1}$$

when $f(x) = (mx+c)^n$.

Evaluating limits when $x \to a$.

1. Show $\lim_{x\to 1}(6x^2 - 4x + 3) = 5$.
2. Show $\lim_{x\to 7} \frac{x^2 - 49}{x - 7} = 14$.
3. Show $\lim_{x\to 2} \frac{x^2 - 6x + 8}{x - 2} = -2$.
4. Show $\lim_{x\to -5} \frac{2x^2 + 9x - 5}{x + 5} = -11$.
5. Show $\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = 3$.
6. Show $\lim_{x\to 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3} = 1/2$.
7. Show $\lim_{x\to -2} \frac{x^3 + 8}{x + 2} = 4$.
8. Show $\lim_{x\to 3} \frac{x^4 - 81}{x - 3} = 108$.
9. Show $\lim_{x\to 5} \frac{x^5 - 3125}{x - 5} = 3125$.
10. Show $\lim_{x\to a} \frac{x^{12} - a^{12}}{x - a} = 12a^{11}$.
11. Show $\lim_{x\to a} \frac{x^{5/2} - a^{5/2}}{x - a} = (5/2)a^{3/2}$.
12. Show $\lim_{x\to a} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{x - a} = (5/3)(a+2)^{2/3}$.
13. Show $\lim_{x\to 4} \frac{x^3 - 64}{x^2 - 16} = 6$.
14. Show $\lim_{x\to 2} \frac{x^5 - 32}{x^3 - 8} = 20/3$.
15. Show $\lim_{x\to 1} \frac{x^n - 1}{x - 1} = n$.
16. Show $\lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \frac{1}{2\sqrt{a}}$.
17. Show $\lim_{x\to 2} \frac{\sqrt{3 - x} - 1}{2 - x} = 1/2$.
18. Show $\lim_{x\to a} \frac{\sqrt{a + 2x} - \sqrt{3x}}{\sqrt{3a + x} - 2\sqrt{x}} = \frac{2\sqrt{3}}{9}$.
19. Show $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.

Evaluating limits when $x \to \infty$.

1. Show $\lim_{x\to\infty} \frac{x+2}{x-2} = 1$.

2. Show $\lim_{x\to\infty} \frac{3x^2+2x-5}{5x^2+3x+1} = 3/5$.

3. Show $\lim_{x\to\infty} \frac{x^2-7x+11}{3x^2+10} = 1/3$.

4. Show $\lim_{x\to\infty} \frac{2x^3-5x+7}{7x^3+2x^2-6} = 2/7$.

5. Show $\lim_{x\to\infty} \frac{(3x-1)(4x-5)}{(x+6)(x-3)} = 12$.

6. Show $\lim_{x\to\infty} \frac{x}{\sqrt{4x^2+1}-1} = 1/2$.

7. Show $\lim_{x\to-\infty} 2^x = 0$.

8. Show $\lim_{t\to\infty} \frac{t+1}{t^2+1} = 0$.

9. Show $\lim_{n\to\infty} \sqrt{n^2+1} - n = 0$.

10. Show $\lim_{n\to\infty} \sqrt{n^2+n} - n = 1/2$.