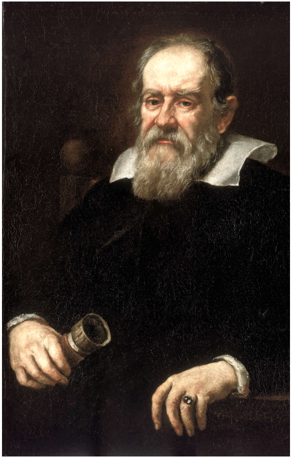
Modeling Rates of Change: Introduction to the Issues

The Legacy of Galileo, Newton, and Leibniz

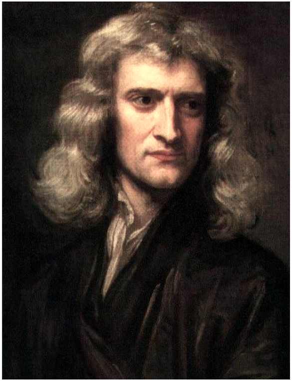


Galileo Galilei (1564-1642)
was interested in falling bodies.
He forged a new scientific methodology:

observe nature,
experiment to test what you observe, and
construct theories that explain the
observations.

The Legacy of Galileo, Newton, and Leibniz

Galileo (1564-1642): Experiment, then draw conclusions.



Sir Isaac Newton (1642-1727)
using his new tools of calculus, explained mathematically why an object, falling under the influence of gravity, will have constant acceleration of $9.8 m/sec^2$.

His laws of motion unified

Newton's laws of falling bodies,

Kepler's laws of planetary motion,

the motion of a simple pendulum,

and virtually every other instance of dynamic motion observed in the universe.

The Legacy of Galileo, Newton, and Leibniz

Galileo (1564-1642): Experiment, then draw conclusions.
Newton (1642-1727): Invented/used calculus to explain motion

Gottfried Wilhelm Leibniz (1646-1716) independently co-invented calculus, taking a slightly different point of view ("infinitesimal calculus") but also studied rates of change in a general setting.

We take a lot of our notation from Leibniz.

The Legacy of Galileo, Newton, and Leibniz

Newton's Question:
How do we find the velocity of a moving object at time t?

What in fact do we mean by *velocity* of the object at the instant of time t? It's easy to find the *average* velocity of an object during a time interval $[t_1, t_2]$, but what is meant by *instantaneous* velocity?

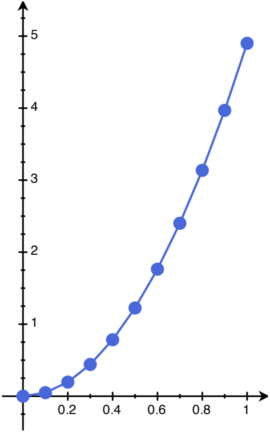
Drop a ball from the top of a building...

At time t, how far has the ball fallen?

Drop a ball from the top of a building...

At time t, how far has the ball fallen? Measure it!

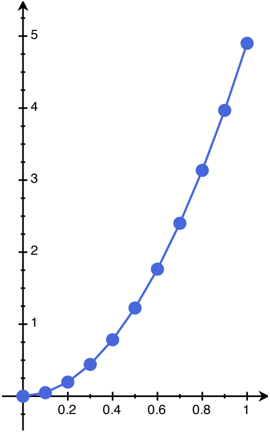
time (s)	distance (m)
0.10	0.049
0.20	0.196
0.30	0.441
0.40	0.784
0.50	1.225
0.60	1.764
0.70	2.401
0.80	3.136
0.90	3.969
1.00	4.900



Drop a ball from the top of a building...

At time t, how far has the ball fallen? Measure it!

time (s)	distance (m)
0.10	0.049
0.20	0.196
0.30	0.441
0.40	0.784
0.50	1.225
0.60	1.764
0.70	2.401
0.80	3.136
0.90	3.969
1.00	4.900

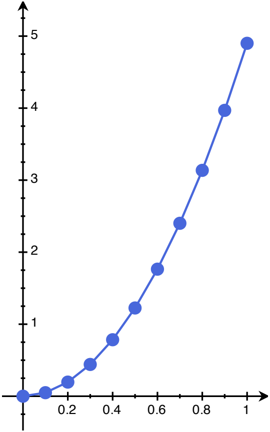


How fast is the ball falling at time t?

Drop a ball from the top of a building...

At time t, how far has the ball fallen? Measure it!

time (s)	distance (m)
0.10	0.049
0.20	0.196
0.30	0.441
0.40	0.784
0.50	1.225
0.60	1.764
0.70	2.401
0.80	3.136
0.90	3.969
1.00	4.900



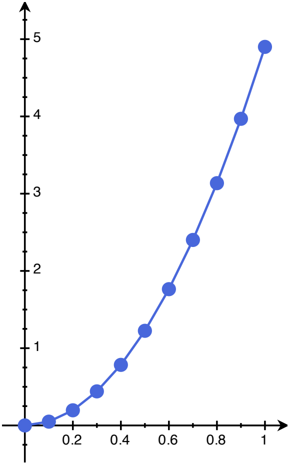
How fast is the ball falling at time t? A little trickier...

Average Speed

Definition

The **average velocity** from $t = t_1$ to $t = t_2$ is

$$\text{avg velocity} = \frac{\text{change in distance}}{\text{change in time}}$$

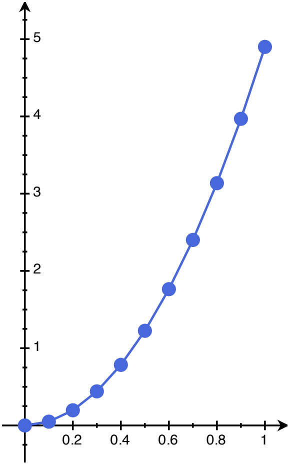


Average Speed

Definition

The **average velocity** from $t = t_1$ to $t = t_2$ is

$$\text{avg velocity} = \frac{\text{change in distance}}{\text{change in time}}$$
$$= \text{slope of secant line}$$

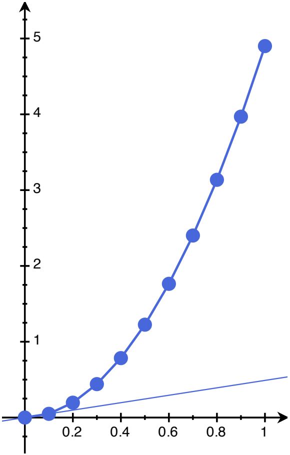


Average Speed

Definition

The **average velocity** from $t = t_1$ to $t = t_2$ is

$$\text{avg velocity} = \frac{\text{change in distance}}{\text{change in time}}$$
$$= \text{slope of secant line}$$

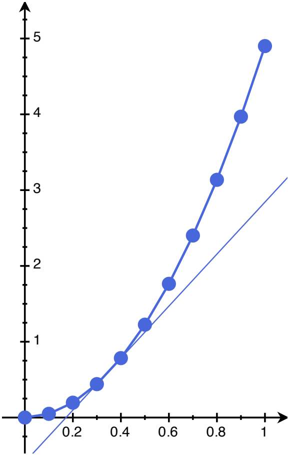


Average Speed

Definition

The **average velocity** from $t = t_1$ to $t = t_2$ is

$$\text{avg velocity} = \frac{\text{change in distance}}{\text{change in time}}$$
$$= \text{slope of secant line}$$

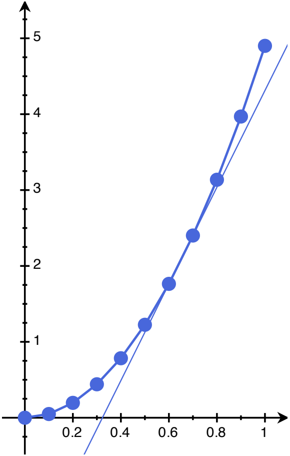


Average Speed

Definition

The **average velocity** from $t = t_1$ to $t = t_2$ is

$$\text{avg velocity} = \frac{\text{change in distance}}{\text{change in time}}$$
$$= \text{slope of secant line}$$

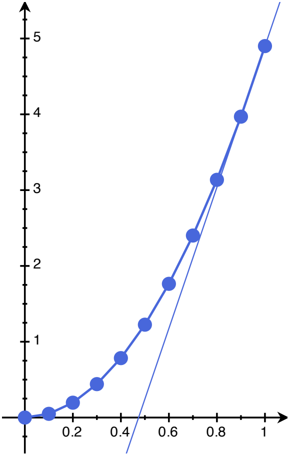


Average Speed

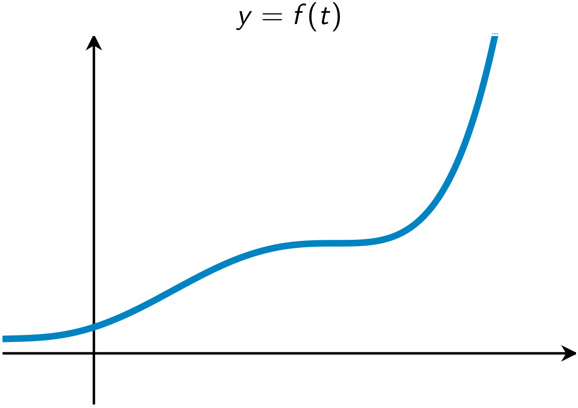
Definition

The **average velocity** from $t = t_1$ to $t = t_2$ is

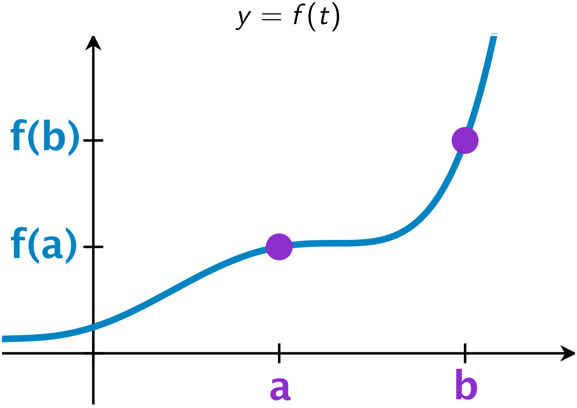
$$\text{avg velocity} = \frac{\text{change in distance}}{\text{change in time}}$$
$$= \text{slope of secant line}$$



Plot position f versus time t:



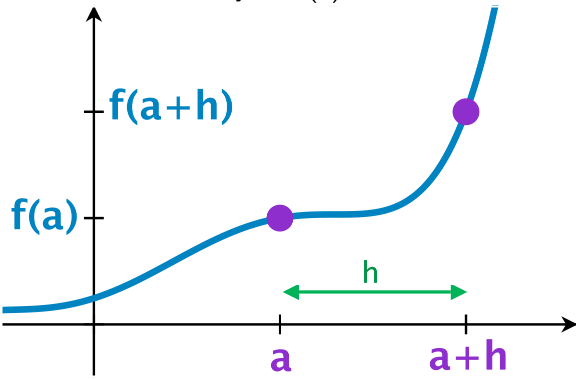
Plot position f versus time t:



Pick two points on the curve $(a, f(a))$ and $(b, f(b))$.

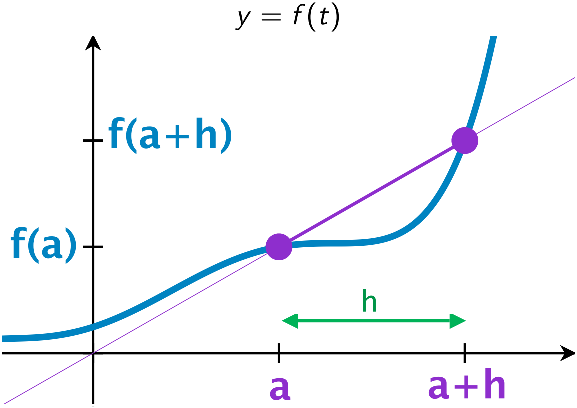
Plot position f versus time t:

$y = f(t)$



Pick two points on the curve $(a, f(a))$ and $(b, f(b))$. Rewrite $b = a + h$.

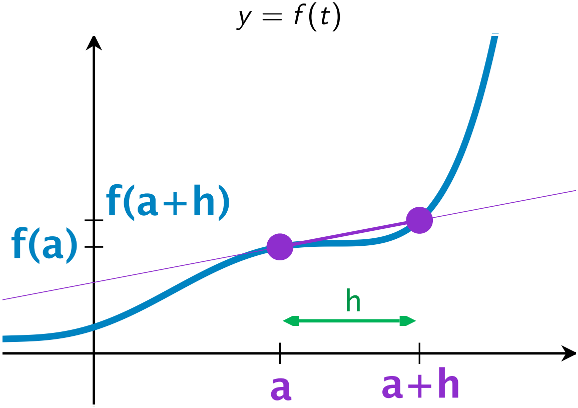
Plot position f versus time t:



Pick two points on the curve $(a, f(a))$ and $(b, f(b))$. Rewrite $b = a + h$.
Slope of the line connecting them:

$$\text{avg velocity} = m = \frac{f(a+h) - f(a)}{h} \qquad \text{“difference quotient”}$$

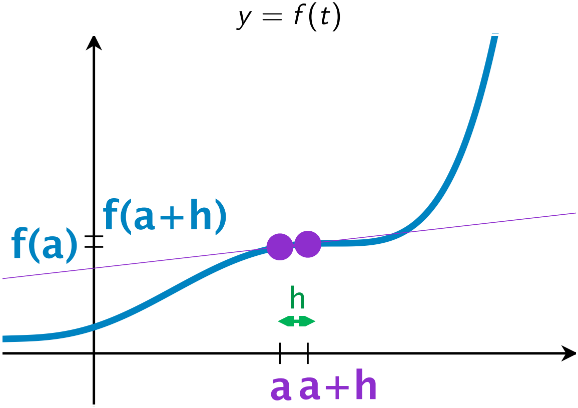
Plot position f versus time t:



Pick two points on the curve $(a, f(a))$ and $(b, f(b))$. Rewrite $b = a + h$.
Slope of the line connecting them:

$$\text{avg velocity} = m = \frac{f(a+h) - f(a)}{h} \qquad \text{“difference quotient”}$$

Plot position f versus time t:

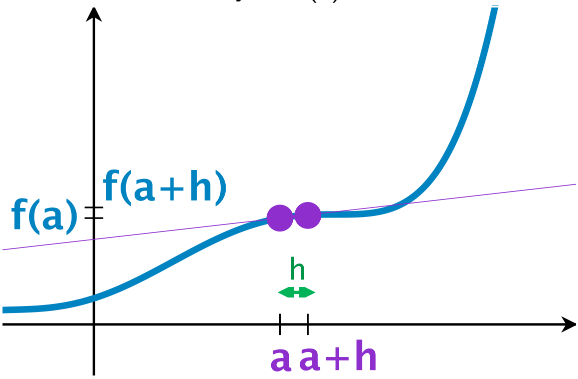


Pick two points on the curve $(a, f(a))$ and $(b, f(b))$. Rewrite $b = a + h$.
Slope of the line connecting them:

$$\text{avg velocity} = m = \frac{f(a+h) - f(a)}{h} \qquad \text{“difference quotient”}$$

Plot position f versus time t:

$y = f(t)$



Pick two points on the curve $(a, f(a))$ and $(b, f(b))$. Rewrite $b = a + h$.
Slope of the line connecting them:

$$\text{avg velocity} = m = \frac{f(a+h) - f(a)}{h} \qquad \text{“difference quotient”}$$

The smaller h is, the more useful m is!

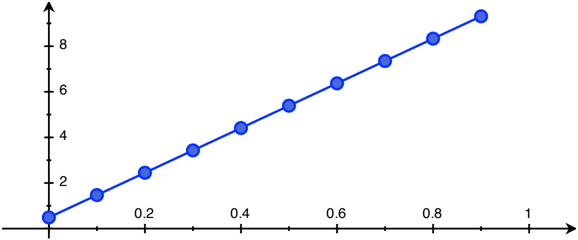
Derived Table of Velocities and Accelerations $[t, t + 0.1]$

(See applet or spreadsheets)

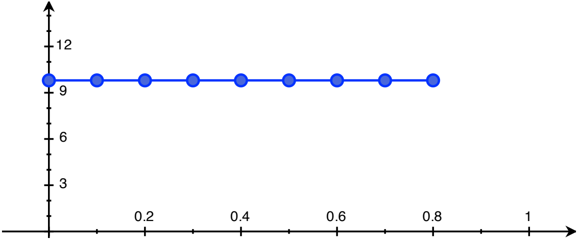
time (s)	distance (m)	speed (m/s)	acc (m/s/s)
0.10	0.049	1.470000	9.800000
0.20	0.196	2.450000	9.800000
0.30	0.441	3.430000	9.800000
0.40	0.784	4.410000	9.800000
0.50	1.225	5.390000	9.800000
0.60	1.764	6.370000	9.800000
0.70	2.401	7.350000	9.800000
0.80	3.136	8.330000	9.800000
0.90	3.969	9.310000	9.800000
1.00	4.900	10.290000	9.800000

Graphs of Velocities and Accelerations $[t, t + 0.1]$

Velocity $y = v(t)$



Acceleration $y = a(t)$



More about the falling ball problem

Suppose the speed of a falling object is given by the function $v(t)$. Then the average acceleration over the interval $[t, t+h]$ is given by the quotient

$$\text{avg acceleration} = \frac{v(t+h) - v(t)}{h}.$$

More about the falling ball problem

Suppose the speed of a falling object is given by the function $v(t)$. Then the average acceleration over the interval $[t, t+h]$ is given by the quotient

$$\text{avg acceleration} = \frac{v(t+h) - v(t)}{h}.$$

Hypotheses:

1. The acceleration of a falling object is constant as a function of time.
2. The speed of a falling object is linear as a function of time.

More about the falling ball problem

Suppose the speed of a falling object is given by the function $v(t)$. Then the average acceleration over the interval $[t, t+h]$ is given by the quotient

$$\text{avg acceleration} = \frac{v(t+h) - v(t)}{h}.$$

Hypotheses:

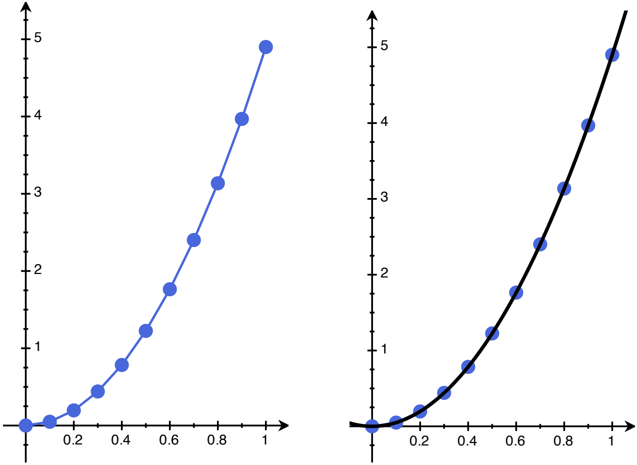
1. The acceleration of a falling object is constant as a function of time.
2. The speed of a falling object is linear as a function of time.

Open Questions

1. Can we find a description (i.e., a formula) for the distance function?
2. How can we get better approximations to the instantaneous velocities?

For now, we cheat...

Ask a computer to fit the curve to these points:



$$f(t) \approx 4.9t^2$$

For now, we cheat...

Ask a computer to fit the curve to these points:

$$\boxed{f(t) \approx 4.9t^2}$$

Or! Since $v(0) = 0$ and $f(0) = 0$, linear velocity means

$$\text{instantaneous velocity} = \text{average velocity} = a * t,$$

which in turn means

$$f(t) = \frac{1}{2} * v(t) * t = \frac{1}{2} * at * t = \frac{1}{2} * 9.8t * t = 4.9t^2$$

Goal: Rates of change in general

Think: $f(x)$ is

distance versus time x, or
profit versus production volume x, or
birthrate versus population x, or...

Goal: Rates of change in general

Think: $f(x)$ is

distance versus time x, or
profit versus production volume x, or
birthrate versus population x, or...

Definition

Given a function f, the *average rate of change* of f over an interval $[x, x+h]$ is

$$\frac{f(x+h)-f(x)}{h}.$$

The average rate of change is also what we have called *the difference quotient* over the interval.

Goal: Rates of change in general

Think: $f(x)$ is

distance versus time x, or
profit versus production volume x, or
birthrate versus population x, or...

Definition

Given a function f, the *average rate of change* of f over an interval $[x, x+h]$ is

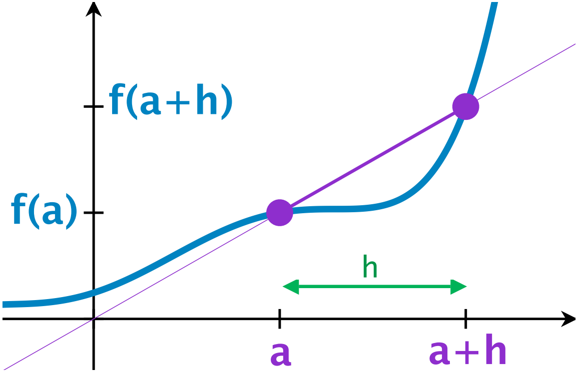
$$\frac{f(x+h) - f(x)}{h}.$$

The average rate of change is also what we have called *the difference quotient* over the interval.

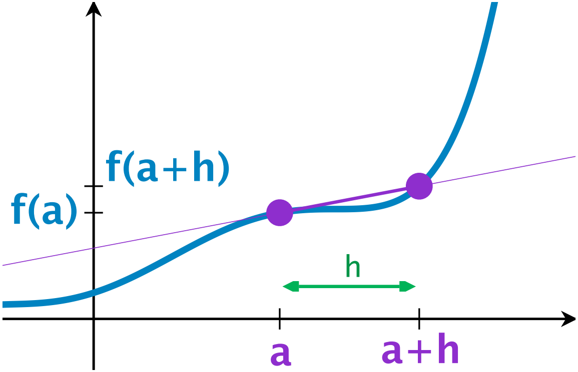
Definition

The *instantaneous rate of change* of a function at a point x is the limit of the average rates of change over intervals $[x, x+h]$ as $h \to 0$.

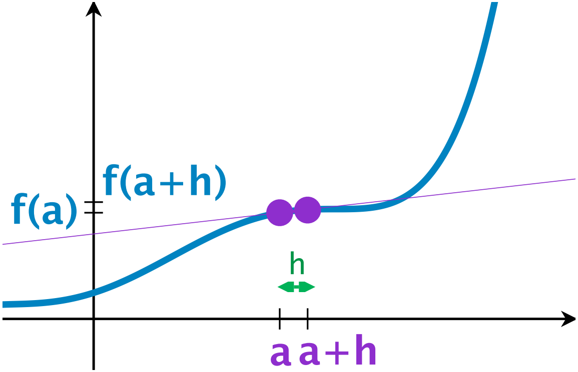
Average rate of change → Instantaneous rate of change



Average rate of change → Instantaneous rate of change



Average rate of change → Instantaneous rate of change



Instantaneous Rate of Change for e^x

(See the applet or the spreadsheets)

Future goals:

1. Get good at limits.
2. Explore instantaneous rates of change further, as limits of difference quotients.
3. Explore the geometric meaning of the definition of instantaneous rate of change at a point.
4. Apply the definition to each of the elementary functions to see if there are formula-like rules for calculating the instantaneous rate of change.
5. Use the definition of instantaneous rate of change and its consequences to obtain explicit functions for the position, velocity, and acceleration of a falling object.