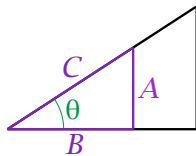
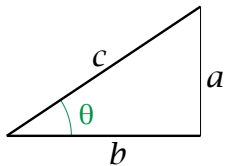


Trigonometric functions

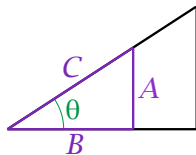
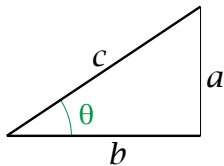
Step one: similar triangles



Two similar triangles have the same set of angles, and have the properties that

$$\frac{A}{B} = \frac{a}{b}, \quad \frac{B}{C} = \frac{b}{c}, \quad \text{and} \quad \frac{A}{C} = \frac{a}{c}.$$

Step one: similar triangles



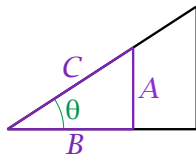
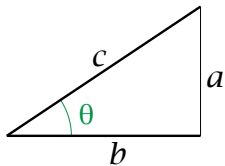
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Define

$$\cos(\theta) = \frac{b}{c} \quad \text{and} \quad \sin(\theta) = \frac{a}{c}.$$

Step one: similar triangles



Two similar triangles have the same set of angles, and have the properties that

$$\frac{A}{B} = \frac{a}{b}, \quad \frac{B}{C} = \frac{b}{c}, \quad \text{and} \quad \frac{A}{C} = \frac{a}{c}.$$

Define

$$\cos(\theta) = \frac{b}{c} \quad \text{and} \quad \sin(\theta) = \frac{a}{c}.$$

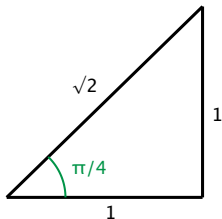
Then let

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{a}{b}, \quad \sec(\theta) = \frac{1}{\sin(\theta)} = \frac{c}{a},$$

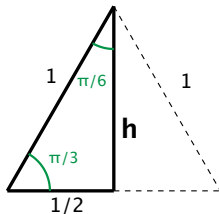
$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{c}{a}, \quad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{b}{a}.$$

Easy angles:

isosceles right triangle:



equilateral triangle cut in half:

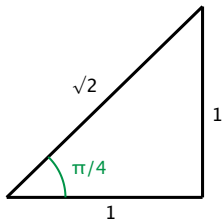


$$h = \sqrt{1 - (1/2)^2} = \sqrt{3}/2$$

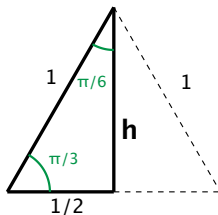
	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)$	$\sec(\theta)$	$\csc(\theta)$	$\cot(\theta)$
$\pi/4$						
$\pi/3$						
$\pi/6$						

Easy angles:

isosceles right triangle:



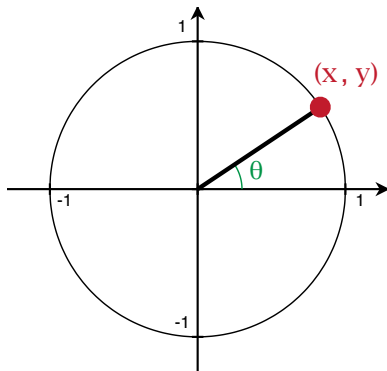
equilateral triangle cut in half:



$$h = \sqrt{1 - (1/2)^2} = \sqrt{3}/2$$

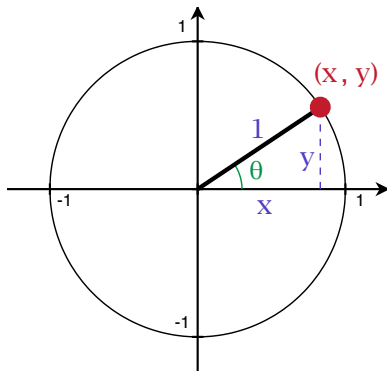
	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)$	$\sec(\theta)$	$\csc(\theta)$	$\cot(\theta)$
$\pi/4$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\pi/3$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	2	$\frac{2}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
$\pi/6$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	2	$\sqrt{3}$

Step two: the unit circle



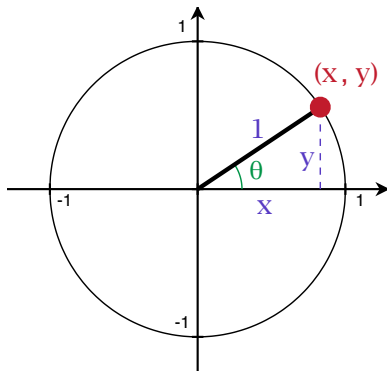
For $0 < \theta < \frac{\pi}{2} \dots$

Step two: the unit circle



For $0 < \theta < \frac{\pi}{2} \dots$

Step two: the unit circle



For $0 < \theta < \frac{\pi}{2} \dots$

$$\cos(\theta) = \frac{x}{1} = x$$

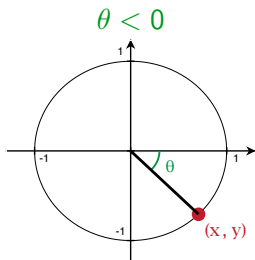
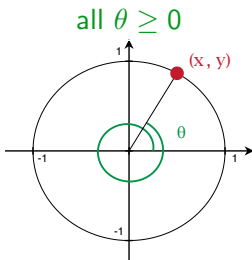
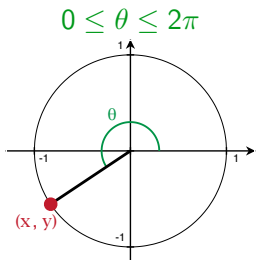
$$\sin(\theta) = \frac{y}{1} = y$$

Use this idea to extend trig functions to any $\theta \dots$

Define

$$\cos(\theta) = x \quad \sin(\theta) = y,$$

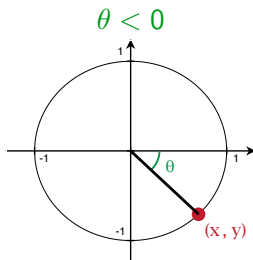
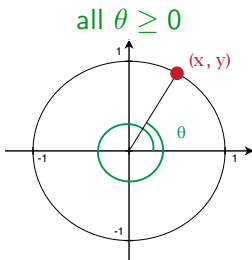
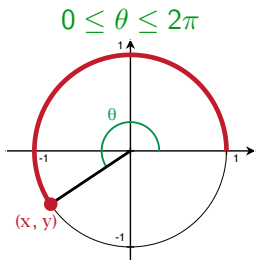
there θ is defined by...



Define

$$\cos(\theta) = x \quad \sin(\theta) = y,$$

there θ is defined by...



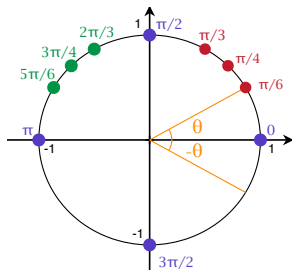
Sidebar: In calculus, radians are king. Where do they come from?

Circumference of a unit circle: 2π

Arclength of a wedge with angle θ :

$$\frac{\theta}{360^\circ} * 2\pi \quad (\text{if in degrees}) \quad \text{or} \quad \frac{\theta}{2\pi} * 2\pi = \boxed{\theta} \quad (\text{if in radians})$$

Reading off of the unit circle



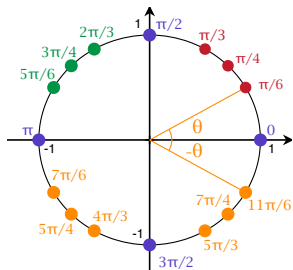
$$\cos(\pi - \theta) = -\cos(\theta) \quad \sin(\pi - \theta) = \sin(\theta)$$

$$\cos(-\theta) = \cos(\theta) \quad \sin(-\theta) = -\sin(\theta)$$

$$\cos(2\pi n + \theta) = \cos(\theta) \quad \sin(2\pi n + \theta) = \sin(\theta)$$

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$			
$\cos(\theta)$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$			
$\sin(\theta)$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$			
	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	
$\cos(\theta)$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$							
$\sin(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$							

Reading off of the unit circle



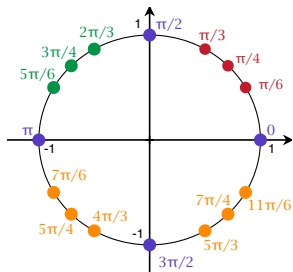
$$\cos(\pi - \theta) = -\cos(\theta) \quad \sin(\pi - \theta) = \sin(\theta)$$

$$\cos(-\theta) = \cos(\theta) \quad \sin(-\theta) = -\sin(\theta)$$

$$\cos(2\pi n + \theta) = \cos(\theta) \quad \sin(2\pi n + \theta) = \sin(\theta)$$

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$			
$\cos(\theta)$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$			
$\sin(\theta)$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$			
	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	
$\cos(\theta)$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	
$\sin(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	

Reading off of the unit circle



$$\cos(\pi - \theta) = -\cos(\theta) \quad \sin(\pi - \theta) = \sin(\theta)$$

$$\cos(-\theta) = \cos(\theta) \quad \sin(-\theta) = -\sin(\theta)$$

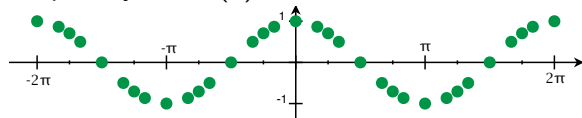
$$\cos(2\pi n + \theta) = \cos(\theta) \quad \sin(2\pi n + \theta) = \sin(\theta)$$

$$x^2 + y^2 = 1 \implies \cos^2(\theta) + \sin^2(\theta) = 1$$

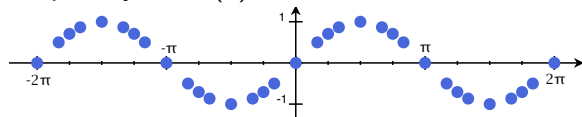
	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$			
$\cos(\theta)$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$			
$\sin(\theta)$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$			
	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	
$\cos(\theta)$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	
$\sin(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	

Plotting on the θ -y axis

Graph of $y = \cos(\theta)$:

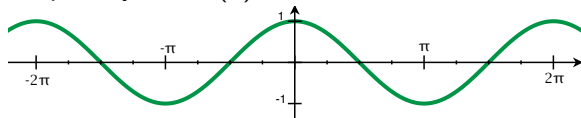


Graph of $y = \sin(\theta)$:

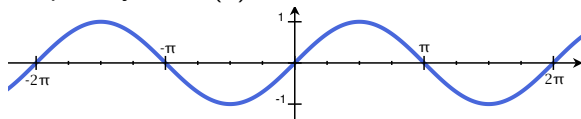


Plotting on the θ - y axis

Graph of $y = \cos(\theta)$:

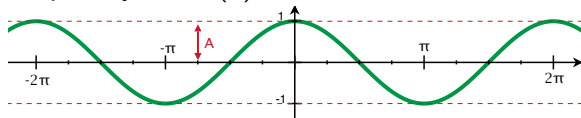


Graph of $y = \sin(\theta)$:



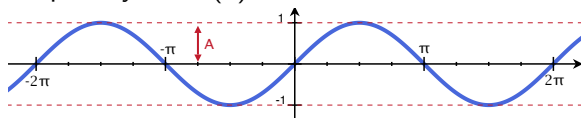
Plotting on the θ -y axis

Graph of $y = \cos(\theta)$:



$$A = \text{Amplitude} = \frac{1}{2} \text{ length of the range} = 1$$

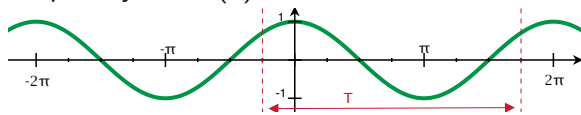
Graph of $y = \sin(\theta)$:



$$A = \text{Amplitude} = \frac{1}{2} \text{ length of the range} = 1$$

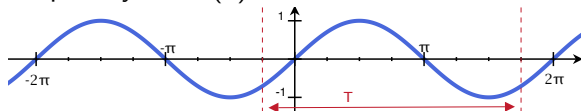
Plotting on the θ -y axis

Graph of $y = \cos(\theta)$:



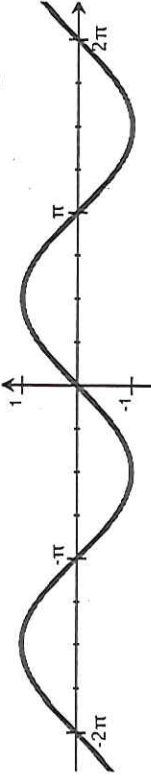
$A = \text{Amplitude} = \frac{1}{2} \text{ length of the range} = 1$
 $T = \text{Period} = \text{time to repeat} = 2\pi$

Graph of $y = \sin(\theta)$:



$A = \text{Amplitude} = \frac{1}{2} \text{ length of the range} = 1$
 $T = \text{Period} = \text{time to repeat} = 2\pi$

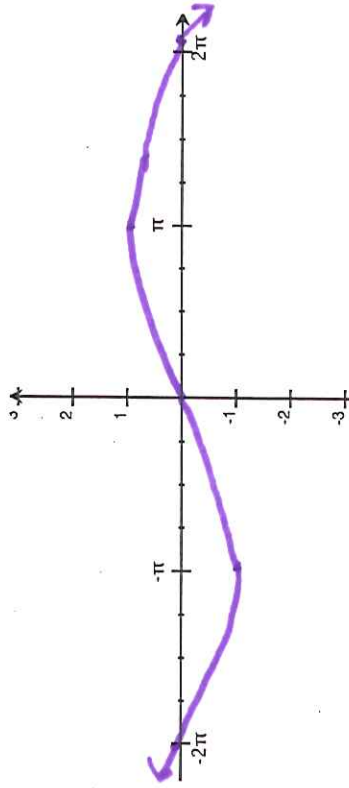
Transform the graph of $\sin(\theta)$ into the graph of $2 \sin(\frac{1}{2}\theta + \frac{\pi}{6}) - 1$:



*

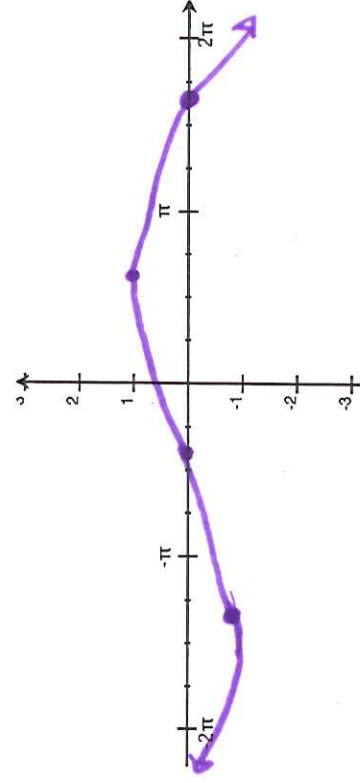
$$\sin\left(\frac{1}{2}\theta\right)$$

Stretch
by 2

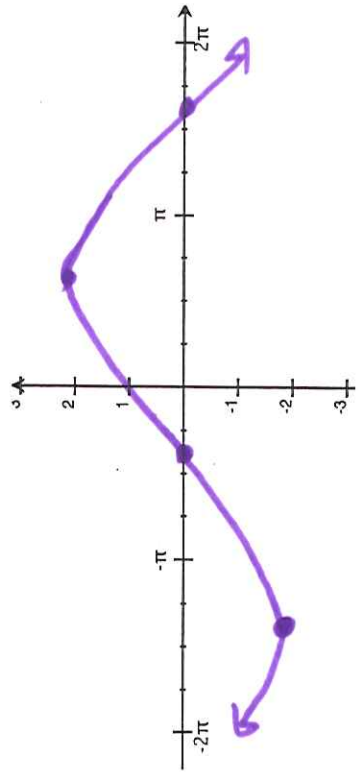


$$\sin\left(\frac{1}{2}\left(\theta + \frac{\pi}{3}\right)\right)$$

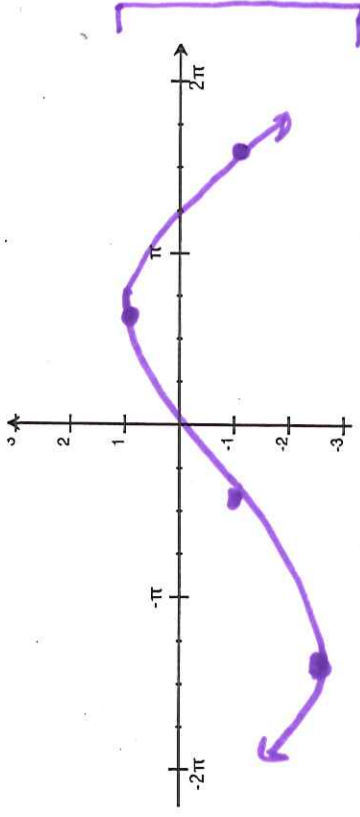
l.f.t
pi/3



$$2 \sin\left(\frac{1}{2}\left(\theta + \frac{\pi}{3}\right)\right)$$



$$2 \sin\left(\frac{1}{2}\left(\theta + \frac{\pi}{3}\right)\right) - 1$$



What is the amplitude of $2 \sin\left(\frac{1}{2}\theta + \frac{\pi}{6}\right) - 1$? 2

What is the period? $4\pi = \frac{1}{1/2} \times 2\pi$

Trig identities to know and love:

Even/odd:

$$\cos(-\theta) = \cos(\theta) \quad (\text{even}) \quad \sin(-\theta) = -\sin(\theta) \quad (\text{odd})$$

Pythagorean identity:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

Angle addition:

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$

(in particular $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$)

Check

$$1 = \cos(0) = \cos(0+0)$$

$$\stackrel{?}{=} \cos(0) \cos(0) - \sin(0) \sin(0)$$

$$= 1 * 1 - 0 * 0 = 1 \quad \checkmark$$

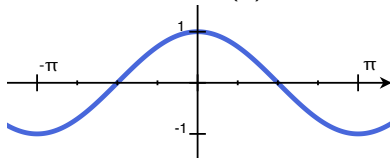
$$0 = \sin(0) = \sin(0+0)$$

$$= \sin(0) \cos(0) + \sin(0) \cos(0)$$

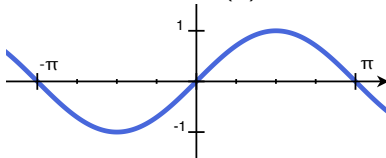
$$= 0 * 1 + 0 * 1 = 0 \quad \checkmark$$

Other trig functions

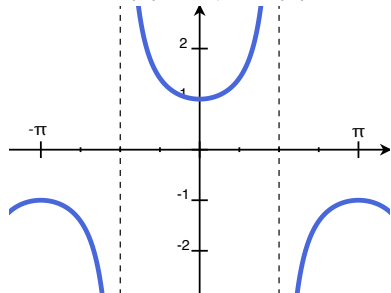
$$y = \cos(\theta)$$



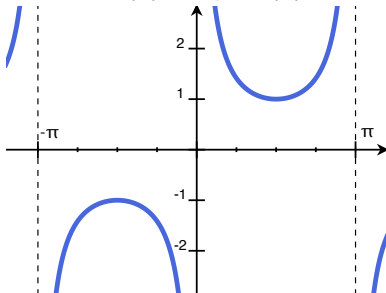
$$y = \sin(\theta)$$



$$\sec(\theta) = 1/\cos(\theta)$$

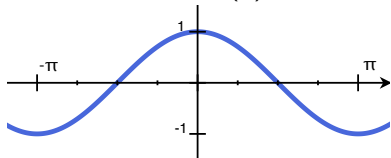


$$\csc(\theta) = 1/\sin(\theta)$$

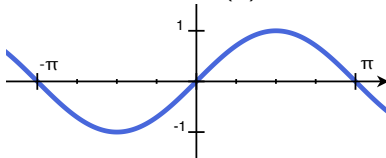


Other trig functions

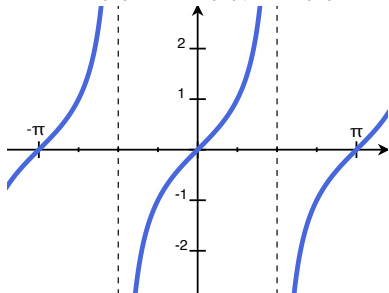
$$y = \cos(\theta)$$



$$y = \sin(\theta)$$



$$\tan(\theta) = \sin(\theta) / \cos(\theta)$$



$$\cot(\theta) = \cos(\theta) / \sin(\theta)$$

