Math 3 Fall 2011

1. Let $f(x) = \frac{2x+1}{3x+2}$. Which of the following is the limit definition of $f'(2)$?

(a) $\lim_{x\to 2} \frac{2x+1}{3x+2}$

(b) $\lim_{h\to 0} \frac{1}{h}\left(\frac{2h+1}{3h+2} - \frac{2\cdot 2+1}{3\cdot 2+2}\right)$

(c) $\lim_{h\to 0} \frac{1}{h}\left(\frac{2x+2h+1}{3x+3h+2} - \frac{2x+1}{3x+2}\right)$

(d) $\lim_{h\to 0} \frac{1}{h}\left(\frac{2(2+h)+1}{3(2+h)+2} - \frac{2\cdot 2+1}{3\cdot 2+2}\right)$

(e) None of the above.

$$\lim_{h\to 0} \frac{1}{h}\left(f(2+h) - f(2)\right)$$

2. Which of the following describes the behavior of $f(x) = \frac{x^2-1}{x-1}$ at the point $x = 1$?

(a) The function f has a jump discontinuity at $x = 1$.

(b) The function f has a continuous extension at $x = 1$.

(c) The function f has a removable discontinuity at $x = 1$.

(d) The function f is continuous on $(-\infty, \infty)$.

(e) None of the above.

$$f(x) = \frac{(x-1)}{(x-1)}(x+1)$$

3. Let $f(x) = x^2 + e^x + \sin(x)$. Which of the following is $f'(x)$?

(a) $f'(x) = 2x + e^x + \cos(x)$

(b) $f'(x) = x^3/3 + e^x - \cos(x)$

(c) $f'(x) = x^3/3 + e^x - \cos(x) + C$

(d) $f'(x) = 2x + e^x + \cos(x) + C$

(e) None of the above.

4. Let $f(x) = \ln\left(\frac{x^2+3}{\sin(x)}\right)$. Which of the following is $f'(x)$?

(a) $f'(x) = \frac{\sin(x)}{x^2+3}$

(b) $f'(x) = \frac{\sin(x)}{x^2+3}\left(\frac{\cos(x)}{2x}\right)$

(c) $f'(x) = \frac{2x\sin(x) - (x^2+3)\cos(x)}{(x^2+3)\sin(x)}$

(d) $f'(x) = 2x\sin(x)\ln\left(\frac{x^2+3}{\sin(x)}\right)$

(e) None of the above

$$f' = \left(1\Big/\frac{x^2+3}{\sin(x)}\right)\cdot\frac{d}{dx}\left(\frac{x^2+3}{\sin(x)}\right)$$

$$= \left(\frac{\cancel{\sin(x)}}{x^2+3}\right)\left(\frac{2x\sin(x) - (x^2+3)\cos(x)}{\sin^2(x)}\right)$$

5. Let

$$f(x) = \begin{cases} |x| & \text{when } x \neq 0, \\ 1 & \text{when } x = 0. \end{cases}$$

Which of the following is true?

(a) The function f is not continuous at $x = 0$, because $\lim_{x \to 0} f(x)$ does not exist.

(b) The function f is not continuous at $x = 0$, because the function f is not defined at $x = 0$.

(c) The function f is not continuous at $x = 0$, because $f(0) \neq \lim_{x \to 0} f(x)$.

(d) The function f is not continuous at $x = 0$, because $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist.

(e) None of the above.

6. Which of the following is the equation of the tangent line to $f(x) = \sec(x) + 2$ through the point $(\pi/4, 2 + \sqrt{2})$?

(a) $y = (\tan(x))x + (2 + \sqrt{2})$.

(b) $y = x + 2 + \sqrt{2}$

(c) $y = \sec(x)\tan(x)(x - \pi/4) + 2 + \sqrt{2}$

(d) $y = \sqrt{2}(x - \pi/4) + 2 + \sqrt{2}$

(e) None of the above.

$f' = \sec(x)\tan(x) = \sin(x)/\cos^2(x)$

$f'(\pi/4) = (1/\sqrt{2})/(1/\sqrt{2})^2 = \sqrt{2}$

$f(\pi/4) = \sqrt{2} + 2$

7. Let $f(x) = x^2 + x + 1$. Find $c \in (0, 2)$ so that

$$f'(c) = \frac{f(2) - f(0)}{2}.$$

(a) $c = 0$

(b) $c = 3/4$

(c) $c = 1$

(d) $c = 5/4$

(e) None of the above.

$f' = 2x + 1 \quad f(2) = 4 + 2 + 1 = 7, \; f(0) = 1$

$2c + 1 = \frac{7-1}{2} = 3$

$c = \frac{3-1}{2} = 1$

8. What is the maximum value of the function $f(x) = x^3 - 3x + 2$ on the interval $[1/2, 4]$?

(a) 1

(b) 0

(c) 54

(d) 5/8

(e) None of the above.

$f' = 3x^2 - 3 = 3(x^2 - 1)$

$= 3(x+1)(x-1)$ $\quad x = \pm 1$

-1 not in $[1/2, 4]$

$1/2$	$\frac{1}{8} - \frac{3}{2} + 2 = \frac{1 - 12 + 16}{8} = 5/8$
4	$4(\underbrace{16-3}_{13}) + 2 = 54$
1	0

f': $[1/2 \; - \; 1 \; + \; 4]$

$\searrow \; \nearrow$

9. Suppose that polonium decays at a rate proportional to the amount present. If a sample of polonium-210 decays so that there is 50 g left after 140 days and 25 g left after 280 days, how much polonium did the sample have to begin with?

(a) $50(e^{140/280})$ g
(b) $100e^0$ g
(c) $100e^{210/280}$ g
(d) $(50/25)e^{140/280}$ g
(e) None of the above

$y = Ae^{kt}$

$50 = Ae^{k140}$

$25 = Ae^{k280}$

divide: $2 = e^{k(140-280)} = e^{k(-140)}$

$K = -\frac{1}{140}\ln(2)$

so $50 = Ae^{-\ln(2)} = A(\frac{1}{2}) \to A = 100$

10. Solve

$$\frac{dy}{dx} = 3x^2y^2.$$

(a) $y = 3(6x)y^2 + 3(x^2)\left(\frac{dy}{dx}\right)$
(b) $y = \frac{1}{C - x^3}$
(c) $y = \ln\left(\sqrt[3]{x^3 + C}\right)$
(d) $y = \frac{1}{x} + C$
(e) None of the above

$-\frac{1}{y} = \int \frac{1}{y^2}\,dy = \int 3x^2\,dx = x^3 + c$

$y = -\frac{1}{x^3 + c} = \frac{1}{-x^3 \underbrace{-c}_{\text{arbitrary}}} = \frac{1}{-x^3 + D}$

11. Suppose that

$$\frac{dy}{dx} = x + 3y;\ y(0) = 1$$

Use Euler's method to approximate $y(1)$ with step size $1/2$.

(a) $1 + 3(1/2)$
(b) $13/2$
(c) $1 - \frac{1}{0 + 3(1)}$
(d) $1/3 + 1/2 - 1.$
(e) None of the above.

x_i	y_i	m_i
0	1	$0 + 3 = 3$
$1/2$	$1 + 3/2 = 5/2$	$\frac{1}{2} + 3 \cdot 5/2 = \frac{16}{2} = 8$
1	$5/2 + \frac{1}{2} \cdot 8 = 13/2$	

12. Differentiate $y = x^{x^3}$.

(a) $y' = (x^2 + 3x^2\ln(x))\,x^{x^3}$
(b) $y' = x^3(x^{x^3-1})$
(c) $y' = 3x^2(x^{x^3})$
(d) $y' = x^2(3 + \ln(x))$
(e) None of the above.

"logarithmic differentiation"

$\ln(y) = \ln(x^{x^3})$

$= x^3\ln(x)$

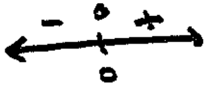
$\frac{1}{y}y' = 3x^2\ln(x) + x^2$

$y' = (3x^2\ln(x) + x^2)\,y$

13. Let $f(x) = x^4 + x^2 - 3$. Which of the following describes f in a small interval around $x = 1$?

(a) f is positive, concave up, and increasing.
(b) f is negative, concave up, and increasing.
(c) f is negative, concave down, and decreasing.
(d) f is stationary.
(e) None of the above.

$f' = 4x^3 + 2x = 2x(x^2+1)$



$f'' = 12x^2 + 2 > 0$ for all x

@ $x=1$, $f(1) = 1+1-3 = -1$ neg

$f'(1) = 2(2) = 4$ pos ↗

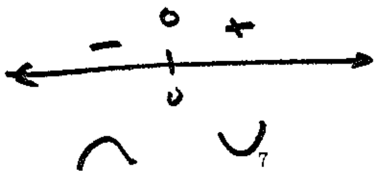
$f''(1) > 0$ ∪

14. Let $f(x) = x^5/20 + x^3/6 + 1$. What are the inflection points of f?

(a) $x = -1$
(b) $x = 0$ and $x = -1$
(c) $x = 0$
(d) The function has no inflection points.
(e) None of the above.

$f' = \frac{x^4}{4} + \frac{x^2}{2}$

$f'' = x^3 + x = x(x^2+1)$



15. Let $f(x) = e^{x^2-x+1}$. What is the linearization of f through the point $(1, e)$?

(a) $L(x) = (2x-1)e^{x^2-x+1}(x-1) + e$
(b) $L(x) = e^{x-1} + e$.
(c) $L(x) = x - 1 + e$
(d) $L(x) = ex$
(e) None of the above

$f' = (2x-1)e^{x^2-x+1}$

$f'(1) = (2-1)e^{1-1+1} = e$

$L(x) = e(x-1) + e$

$= ex$

16. Suppose that the side lengths of an equilateral triangle are shrinking at a rate of 1 in/s, so that the figure is always an equilateral triangle. At the moment when the area of the triangle is $400\sqrt{3}$, at what rate is the area of the triangle changing? (You may use the fact that an equilateral triangle of side length s has area $\frac{\sqrt{3}s^2}{4}$.

(a) $-\frac{\sqrt{3}}{2}$ in^2/s.
(b) $-\frac{\sqrt{3}}{40}$ in^2/s.
(c) $-20\sqrt{3}$ in^2/s.
(d) $-4(400)^2\frac{\sqrt{3}}{3}$ in^2/s.
(e) None of the above.

$h^2 + \left(\frac{l}{2}\right)^2 = l^2$

$h = \frac{\sqrt{3}}{2} l$

$\frac{dl}{dt} = -1$

$A = \frac{1}{2} b \cdot h = \frac{1}{2} \cdot l \cdot \frac{\sqrt{3}}{2} \cdot l$

$= \frac{\sqrt{3}}{4} l^2$

if

$400\sqrt{3} = A = \frac{\sqrt{3}}{4} l^2$

$l^2 = 1600$

$l = 40$

$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \cdot 2 \cdot l \cdot \frac{dl}{dt}$

$= -\frac{\sqrt{3}}{2} l = -\frac{\sqrt{3}}{2} \cdot 40$

17. Suppose that $2x\sin(y) = e^{xy}$. Find a formula for $\frac{dy}{dx}$ in terms of y and x.

(a) $\frac{dy}{dx} = \frac{e^x}{2\cos(y)}$

(b) $\frac{dy}{dx} = \frac{ye^{xy} + x\left(\frac{dy}{dx}\right)}{2\cos(y)}$

(c) $\frac{dy}{dx} = (xy)e^{xy} - 2\cos(y)$

(d) $\frac{dy}{dx} = \frac{ye^{xy} - 2\sin(y)}{-xe^{xy} + 2x\cos(y)}$

(e) None of the above

$2(x\cos(y)y' + \sin(y)) = e^{xy}(y + xy')$

$y'(2x\cos(y) - e^{xy}\cdot x) = ye^{xy} - 2\sin(y)$

18. Compute $\int x^2 + 47x + e^x - \frac{1}{x}dx$.

(a) $x^3/3 + 47x^2/2 + e^x - \ln|x| + C$.

(b) $2x + 47 + e^x + 1/x^2 + C$

(c) $2x + 47 + e^x + 1/x^2$

(d) $x^3/3 + 47x^2/2 + e^x + \ln|x| + C$.

(e) None of the above

19. Which of the following is a formula for the Right Riemann sum of $f(x) = \sin(x)$ on the interval $[0,1]$ with n subintervals of equal length?

(a) $\sum_{i=1}^{n} \frac{1}{n}\sin\left(\frac{i}{n}\right)$

(b) $\sum_{i=0}^{n-1} \frac{i+1}{n}\sin\left(\frac{i+1}{n}\right)$

(c) $\sum_{i=1}^{n-1} \frac{1}{n}\sin\left(\frac{i-1}{n}\right)$

(d) $\sum_{i=1}^{n} \frac{1}{n}\sin\left(\frac{1}{n}\right)$

(e) None of the above.

$\Delta x = \frac{1}{n}$

$x_i = 0 + i\frac{1}{n}$

start @ x_1

$\sum \sin(i/n) * \frac{1}{n}$

20. Compute $\frac{d}{dx}\arctan(e^x)$.

(a) $\sec^2(e^x)(e^x)$

(b) $\frac{e^x}{1+e^{2x}}$

(c) $e^x\tan(e^x)$

(d) $\frac{\sec^2(e^x)}{e^x}$

(e) None of the above

$\frac{1}{1+(e^x)^2}\cdot e^x$

21. Compute $\int -17x\cos(x^2)dx$.

(a) $(-17x^2/2)(\sin(x^2)) + C$
(b) $-17\cos(x^2) + (17)x\sin(x^2)(2x) + C$
(c) $(\frac{-17}{2})\sin(x^2) + C$.
(d) $\frac{-17\sin(x^2)}{x} + C$.
(e) None of the above.

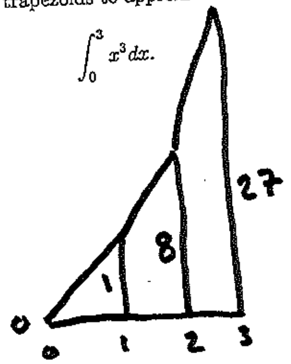
let $u = x^2 \rightarrow du = 2xdx \rightarrow \frac{1}{2}du = xdx$

$-\frac{17}{2}\int \cos(u)du = -\frac{17}{2}\sin(x^2) + c$

22. Use the trapezoid rule with $n = 3$ trapezoids to approximate

$$\int_0^3 x^3 dx.$$

(a) $T_3 = (1/2)(0 + 1 + 8 + 27)$
(b) $T_3 = (1/2)(0 + 2 + 16 + 54)$
(c) $T_3 = (1/2)(0 + 2 + 16 + 27)$
(d) $T_3 = \frac{3^4}{4}$
(e) None of the above.



$\frac{1}{2}(0 + 2\cdot 1 + 2\cdot 8 + 27)$

23. Write down the integral that you would need to compute in order to find the arc-length of the curve $y = e^x$ over the interval $[4, 7]$.

(a) $\int_4^7 e^x dx$
(b) $\int_4^7 1 + e^{2x} dx$
(c) $\int_4^7 \sqrt{1 + e^x} dx$
(d) $\int_4^7 \sqrt{1 + e^{2x}} dx$
(e) None of the above.

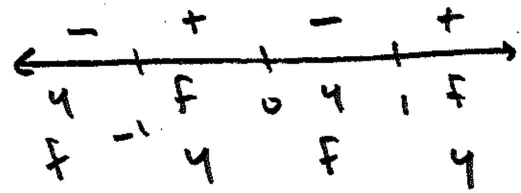
$f' = e^x$

24. Find the area bounded between the curves $f(x) = 20x^3$ and $y = 20x$.

(a) -5
(b) 0
(c) 5
(d) 10
(e) None of the above.

$20x^3 = 20x \rightarrow 0 = x^3 - x$
$= x(x^2 - 1)$
$= x(x+1)(x-1)$

$20(x^3 - x) = f - y$



$20\int_{-1}^{0} x^3 - x dx + 20\int_0^1 x - x^3 dx$

$= 20\left[\left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{x=-1}^{0} - \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{x=0}^{1}\right] = \frac{1}{2}\cdot 20$