Hour Exam #1

Math 3

October 16, 2013

Name (Print): Solutions
Last First

On this, the first of the two Math 3 hour-long exams in Fall 2013, and on the exams that follow, I will work individually, neither giving nor receiving help, guided by the Dartmouth Academic Honor Principle.

Signature:________________________________

Section (circle):

10 (Daugherty) 11 (Daugherty) 12 (Diesel) 2 (Diesel)

Instructions: You are not allowed to use calculators, books, or notes of any kind. All of your answers must be marked on the Scantron form provided or entered on the test, depending on the problem. Take a moment now to print your name and section clearly on your Scantron form and on page 1 of your exam booklet and sign the affirmation. You may write on the exam, but you will only receive credit on Scantron (multiple-choice) problems for what you write on the Scantron form. At the end of the exam, you must turn in both your Scantron form and your exam booklet. There are 15 multiple-choice problems worth 4 points each and 3 long-answer written problems worth a total of 40 points.

Problem	Points	Score
M.C.	60	
16	15	
17	13	
18	12	
Total	100	

1. An object moves horizontally on a number line with its position given by

$$h(t) = -t^2 + 4t + 5.$$

At what t is the acceleration of the object equal to zero?

(a) $t = 2$

(b) $t = -2$

(c) $t = -1$ and $t = 5$

(d) never

(e) None of the above.

$h' = -2t + 4$

$h'' = -2$ ← acceleration never zero!

2. An object that is dropped near the surface of the Planet Doom will fall a distance of $D(t) = 5t^2$ meters in t seconds. What is the average velocity of a falling object during the first 3 seconds after it is dropped?

(a) 10 m/s

(b) 15 m/s

(c) 30 m/s

(d) 45 m/s

(e) None of the above.

$$\frac{D(3) - D(0)}{3 - 0} = \frac{5 \cdot 3^2 - 0}{3}$$

$$= 5 \cdot 3$$

3. Let $f(x) = \tan^2(x^2)$. What is the derivative of $f(x)$?

(a) $2\tan(2x)$

(b) $2\tan(x^2) \cdot \sec^2(x^2) \cdot 2x$

(c) $2\tan(2x)$

(d) $\sec^4(x^2) \cdot 2x$

(e) None of the above.

$f' = 2 \cdot \tan(x^2) \cdot \frac{d}{dx}(\tan(x^2))$

$= 2 \cdot \tan(x^2) \cdot \sec^2(x^2) \cdot 2x$

4. Find the derivative of $y = \log_5(x^2 + 2^x)$.

(a) $\dfrac{1}{x^2 + 2^x} \cdot (2x + x2^{x-1})$

(b) $\dfrac{1}{\ln(5)(2x + 2^x \ln(2))}$

(c) $\dfrac{1}{\ln(5)(x^2 + 2^x)} \cdot (2x + 2^x \ln(2))$

(d) $\dfrac{\ln(5)}{(x^2 + 2^x)} \cdot (2x + 2^x / \ln(2))$

(e) None of the above.

$\frac{d}{dx} a^x = a^x \cdot \ln(a)$

$\frac{d}{dx} \log_a(x) = \dfrac{1}{\ln(a) \cdot x}$

so $\frac{d}{dx} \log_5(x^2 + 2^x)$

$= \dfrac{1}{\ln(5)(x^2 + 2^x)} \cdot \frac{d}{dx}(x^2 + 2^x)$

5. Let $f(x) = \dfrac{(x^2-4)(x+3)}{(x-1)(x+2)}$.
Which of the following accurately describes the graph of $f(x)$?

~~(a)~~ It has a hole at $x = 1$. asymptote

~~(b)~~ It has a horizontal asymptote at $y = 2$. $\lim_{x\to\infty} f = \infty$

~~(c)~~ It has a root at $x = -2$. hole

~~(d)~~ It has a vertical asymptote at $x = -3$. root

(e) None of the above.

6. Consider $f(x) = \frac{x^3-8}{x-2}$. Which of the following best describes this function?

(a) This function is equal to $g(x) = x^2 + 2x + 4$ for all real numbers x.

(b) This function has a continuous extension $\bar{f}(x) = \begin{cases} \frac{x^3-8}{x-2} & x \neq 2 \\ 0 & x = 2. \end{cases}$

(c) This function has continuous extension $\bar{f}(x) = \begin{cases} \frac{x^3-8}{x-2} & x \neq 2 \\ 2 & x = 2. \end{cases}$

(d) This function has continuous extension $\bar{f}(x) = \begin{cases} \frac{x^3-8}{x-2} & x \neq 2 \\ 12 & x = 2. \end{cases}$

(e) This function is not defined at $x = 2$, but has no continuous extension.

$$(x^3 - \underset{\substack{\uparrow \\ 2^3}}{8}) = (x-2)(x^2+2x+4)$$

$$\text{so } \lim_{x\to 2} f(x) = \lim_{x\to 2} x^2+2x+4 = 3\cdot 4 = \boxed{12}$$

7. If $y = \sec(x)$, what is $\frac{d^2y}{dx^2}$?

(a) $\frac{d^2y}{dx^2} = \sec^3(x) + \sec(x)\tan^2 x$

(b) $\frac{d^2y}{dx^2} = \sec^2(x)\tan^2(x)$

(c) $\frac{d^2y}{dx^2} = \sec(x)\tan(x)$

(d) $\frac{d^2y}{dx^2} = \sec(x)$

(e) None of the above.

$$y' = \overset{f}{\sec(x)} \cdot \overset{g}{\tan(x)}$$

So $$y'' = \overset{f'}{\left(\sec(x)\tan(x)\right)}\overset{g}{\tan(x)} + \overset{f}{\sec(x)} \cdot \overset{g'}{\sec^2(x)}$$

$$= \sec(x)\tan^2(x) + \sec^3(x)$$

8. What is the derivative of $f(x) = \arctan(x)$?

(a) $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$

(b) $f'(x) = \dfrac{1}{1+x^2}$

(c) $f'(x) = \dfrac{1}{\sec^2(x)}$

(d) $f'(x) = \text{arcsec}^2(x)$

(e) None of the above.

$$y = \arctan(x)$$

$$\downarrow$$

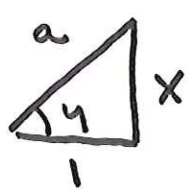
$$\tan(y) = x$$

$$\downarrow d/dx$$

$$\sec^2(y)\, y' = 1 \rightarrow y' = \left(1/\sec(x)\right)^2$$

$$= \left(1/\sqrt{1+x^2}\right)^2$$

$$= \boxed{1/1+x^2}$$



$a^2 = 1^2 + x^2$ so $a = \sqrt{1+x^2}$

so $\sec(y) = a/1$

$= \sqrt{1+x^2}$

9. Suppose $h(x) = f(g(x))$ and you know the following values:

$$f(-4) = 10,\ f'(2) = 6,\ g(1) = 2,\ g'(1) = -4.$$

Find $h'(1)$.

(a) -40
(b) -24
(c) -8
(d) 6
(e) There is not enough information provided.

$$h' = f'(g(x)) \cdot g'(x)$$

so $h'(1) = f'(g(1)) \cdot g'(1)$

$$= f'(2) \cdot (-4)$$

$$= 6 \cdot (-4)$$

10. Suppose $f(x) = x^2 + 1$ and $f(g(x)) = 3 - x$. What is $g(x)$?

(a) $\sqrt{2-x}$
(b) $3 - \sqrt{x-1}$
(c) $\sqrt{3-x} - 1$
(d) $4 - \sqrt{x}$
(e) None of the above.

calculate $f^{-1}(x)$:

$$x = y^2 + 1$$

$$y = \sqrt{x-1}$$

so $g(x) = f^{-1}(f(g(x)))$

$$= \sqrt{(3-x)-1} = \boxed{\sqrt{2-x}}$$

Alternatively: $3 - x = f(g(x)) = g^2 + 1$

so $g^2 = 2 - x$, and $g = \sqrt{2-x}$.

11. If $xy = x^2 + y^3$, what is $\frac{dy}{dx}$?

(a) $\frac{dy}{dx} = \frac{2x + 3y^2}{x}$

(b) $\frac{dy}{dx} = \frac{2x}{x - 3y^2}$

(c) $\frac{dy}{dx} = \frac{2x - y}{x - 3y^2}$

(d) $\frac{dy}{dx} = \frac{2x + 3y^2 - y}{x}$

(e) None of the above.

$y + xy' = 2x + 3y^2y'$

So $y'(x - 3y^2) = 2x - y$

so $y' = \frac{2x-y}{x-3y^2}$

12. Where on the graph of $y^2 = 2x^3 - 16$ is the tangent line vertical?

(a) At the point $(2, 0)$.

(b) At the point $(0, 2)$.

(c) at the points $(0, 4)$ and $(0, -4)$.

(d) Nowhere.

(e) None of the above.

when is y' infinite?

$2yy' = 6x$

$y' = 3x/y$ $\rightarrow$ when $\underline{y=0}$

$0 = 2x^3 - 16$

$x^3 = 8$ $\rightarrow$ $\underline{x=2}$

13. If $f(x) = \dfrac{x^4 - x^2}{x}$ differentiable at $x = 0$?
Choose the best answer.

(a) Yes, because it is an odd function.

(b) Yes, because $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$ both exist and are equal.

(c) No, because $f(x)$ does not have a continuous extension at $x = 0$.

(d) No, because $f(0)$ is not defined. Even though $f(x)$ has the continuous extn $\bar{f}(x) = x^3 - x$.

(e) None of the above.

14. Suppose $f(x)$, $g(x)$, and $h(x)$ are differentiable at $x = a$.
Which of the following statements is **not** always true?

(a) The product $f(x) \cdot g(x) \cdot h(x)$ is differentiable at $x = a$.

(b) $\lim_{x \to a^-} h(x) = \lim_{x \to a^+} h(x)$.

(c) $\frac{d}{dx}(f(x) + g(x) + h(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) + \frac{d}{dx}h(x)$.

(d) $\dfrac{f(x)}{g(x)}$ is differentiable at $x = a$. if $g(a) = 0$.

(e) All of the above statements are always true.

15. Let $f(x)$ be the function graphed to the right. What is the domain of f?

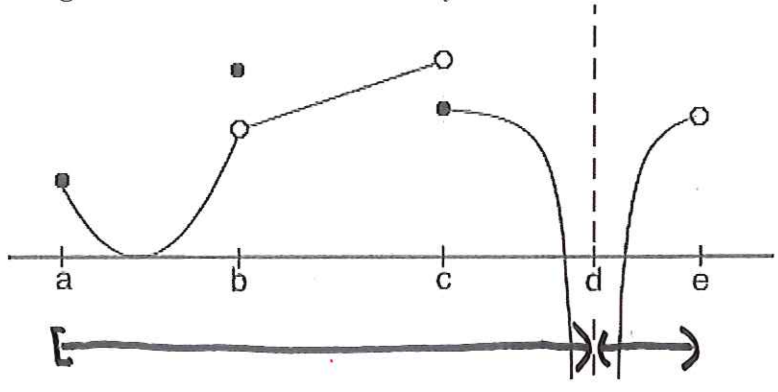
(a) $[a, b) \cup (d, e)$

(b) $(-\infty, d) \cup (d, \infty)$

(c) $[a, b) \cup (b, c) \cup (c, d) \cup (d, e)$

(d) $[a, c) \cup (c, d) \cup (d, e]$

(e) None of the above.



Long answer questions

16. (15 pts) Let $f(x) = \dfrac{2}{x-3}$.

(a) State the limit definition of the derivative of $f(x) = \frac{2}{x-3}$.

$$f'(x) = \lim_{h \to 0} \left(\frac{2}{(x+h)-3} - \frac{2}{x-3} \right) \cdot \frac{1}{h}$$

(b) Calculate the derivative of $f(x)$ using the limit definition of the derivative. Show all of your work and explain your steps to receive any credit.

continuing:

$$f'(x) = 2 \lim_{h \to 0} \frac{(x-3) - (x+h-3)}{(x+h-3)(x-3)h} \quad \text{cancel}$$

$$= 2 \lim_{h \to 0} \frac{-h}{(x+h-3)(x-3)h} \quad \text{cancel}$$

$$= 2 \lim_{h \to 0} \frac{-1}{(x+h-3)(x-3)}$$

$$= 2 \cdot \frac{-1}{(x-3)^2} \quad \text{(plug in } h=0\text{)}$$

(c) Use the derivative rules to calculate the derivative of $f(x)$ to check your answer.

$$f = 2 \cdot (x-3)^{-1} \quad \text{so} \quad f' = 2(-1)(x-3)^{-2} \cdot 1$$

$$= -2/(x-3)^2 \quad \checkmark$$

17. (13 pts) Let $f(x) = e^{\sin(x)}$.

(a) Calculate $f'(\pi/4)$.

$$f'(x) = e^{\sin(x)} \cdot \frac{d}{dx}\sin(x)$$

$$= e^{\sin(x)} \cdot \cos(x)$$

$$\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$$

so $f'(\pi/4) = \boxed{\frac{\sqrt{2}}{2} \cdot e^{\sqrt{2}/2}}$

(b) Find the equation for the line tangent to $y = e^{\sin(x)}$ at $x = \pi/4$.

$$y - y_0 = m(x - x_0)$$

$$\boxed{y - e^{\sqrt{2}/2} = \frac{\sqrt{2}}{2} e^{\sqrt{2}/2}\left(x - \pi/4\right)}$$

or $y = \frac{\sqrt{2}}{2} e^{\sqrt{2}/2} x + e^{\sqrt{2}/2}\left(1 - \frac{\sqrt{2}}{2}\frac{\pi}{4}\right)$

(c) Find all values of x on the interval $[-2\pi, 2\pi]$ where the tangent line to $y = e^{\sin(x)}$ is horizontal.

solve $f' = 0 = \cos(x)\, e^{\sin(x)}$

$e^{\sin(x)}$ never 0

$\cos(x) = 0$

@ $x = n\pi + \frac{\pi}{2}$

in $[-2\pi, 2\pi]$

that leaves $\boxed{-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}}$

18. (12 pts) Let $f(x) = \frac{x-5}{3x-2}$.

(a) Calculate $f^{-1}(x)$.

let $y = f^{-1}(x)$

$x = \frac{y-5}{3y-2} \to (3y-2) \cdot x = y-5$

$3yx - 2x =$

So $y(3x-1) = 2x-5$

$$\boxed{y = \frac{2x-5}{3x-1}}$$

(b) What are the domain and range of $f(x)$?

Domain: where $3x-2 \neq 0$ $\to$ all but $x = \frac{2}{3}$

$(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$

Range (f) = Domain (f^{-1}) : where $3x-1 \neq 0$

$\to$ all but $x = 1/3$

$(-\infty, \frac{1}{3}) \cup (1/3, \infty)$

(c) Describe all vertical and horizontal asymptotes of $f(x)$.

vert: $\lim_{x \to \frac{2}{3}^\pm} f(x) = \pm\infty \Rightarrow$ vert asymp @ $x = 2/3$

horiz: $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x}{3x} = \frac{1}{3}$

$\Rightarrow$ horiz asymp @ $\boxed{y = \frac{1}{3}}$