Name and Section:____________________________

Instructor's Name:____________________________

Math 3 Exam 1

October 17, 2006

Instructions: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- *Print* your name, section number and instructor in the space provided.
- No calculators are allowed.
- Except on multiple choice and true/false questions you must show your work to receive full credit.

Honor Statement:

I have neither given nor received help on this exam, and all of the answers are my own.

____

Signature

Question	Points	Score
1	8	
2	12	
3	15	
4	10	
5	10	
6	10	
7	10	
8	10	
9	5	
Total:	90	

1. Circle True or False. There is no partial credit.

   (a) [2 points] If $f(x)$ is differentiable at a then $f(x)$ is continuous at a.

   A. True B. False

Solution: This is *True.*

   (b) [2 points] If $f(x)$ is continuous at a then $f(x)$ is differentiable at a.

   A. True B. False

Solution: This is *False.*

   (c) [2 points] $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x = a$.

   A. True B. False

Solution: This is *True.*

   (d) [2 points] $\lim_{x \to a} f(x)$ is the slope of the tangent line to $f(x)$ at the point $(a, f(a))$.

   A. True B. False

Solution: This is *False.*

2. Circle the best response. No partial credit will be given.

(a) [3 points] Evaluate

$$\lim_{x\to 2} \frac{x+5}{x^2+3x-10}.$$

A. $-\frac{3}{5}$ B. 0 C. $\frac{8}{5}$ D. Does Not Exist E. None of These

Solution: This limit *D. Does Not Exist*

(b) [3 points] Evaluate

$$\lim_{x\to 0} \frac{\cos(\sin(x))}{\sec(x)}.$$

A. $\frac{\pi}{2}$ B. $\frac{\sqrt{2}}{2}$ C. 1 D. Does Not Exist E. None of These

Solution: This limit is equal to *C. 1* by direct substitution.

(c) [3 points] Evalulate

$$\lim_{x\to \pi/2} \frac{\sin x}{x}.$$

A. $\frac{2}{\pi}$ B. 1 C. $\frac{\sqrt{3}}{2}$ D. Does Not Exist E. None of These

Solution: This limit is equal to *A*. $2/\pi$.

(d) [3 points] Evaluate

$$\lim_{x\to 1} \frac{x^2-1}{|x-1|}.$$

A. 2 B. 0 C. -2 D. Does Not Exist E. None of These

Solution: This limit is equal to *D. Does Not Exist.*

3. Compute the derivatives of the following functions with respect to x. You do not have to simplify your answers. *Show your work.*

   (a) [5 points] Find $f'(x)$ where

$$f(x) = \frac{x^2 + x}{x^3 - 20}$$

.

Solution: Using the quotient rule we find that

$$f'(x) = \frac{(2x+1)(x^3-10) - (x^2+x)(3x^2)}{(x^3-10)^2}.$$

(b) [5 points] Find $h'(x)$ where

$$h(x) = \sqrt{\cos(\sin(x)) + 1}.$$

Solution: We use the chain rule twice to get

$$\begin{aligned} h'(x) &= 1/2(\cos(\sin(x)) + 1)^{-1/2}(-\sin(\sin(x))(\cos(x))) \\ &= \frac{-\sin(\sin(x))\cos(x)}{2\sqrt{\cos(\sin(x)) + 1}}. \end{aligned}$$

(c) [5 points] Let

$$k(x) = \tan\left(h(x)\cos(x)\right)$$

where h is a differentiable function such that $h(\pi) = 0$ and $h'(\pi) = 3$. Find $k'(\pi)$.

Solution: Use the chain rule and product rule to get

$$k'(x) = \sec^2\left(h(x)\cos(x)\right)\left(h'(x)\cos(x) - h(x)\sin(x)\right)$$

and $k'(\pi) = 3$.

4. [10 points] Find the equation of the tangent line to the curve

$$x^2 + y^3 - x^2y^2 = -5$$

through the point $(2, 3)$. Write your solution in the form $y = mx + b$ for some numbers m and b.

Solution: By implicit differentiation with respect to x, we have

$$2x + 3y^2y' - x^2(2yy') - y^2(2x) = 0,$$

and solving for y' we find

$$y' = \frac{y^2(2x) - 2x}{3y^2 - 2x^2y} = \frac{2x(y^2 - 1)}{y(3y - 2x^2)}.$$

So at $(2, 3)$ we have

$$y' = \frac{2 \cdot 2(3^2 - 1)}{3(3 \cdot 3 - 2 \cdot 2^2)} = \frac{32}{3},$$

and so the tangent line through $(2, 3)$ is given by

$$y - 3 = \frac{32}{3}(x - 2).$$

Rearranging to place the answer in the required form gives

$$y = \frac{32}{3}x - \frac{32}{3} \cdot 2 + 3,$$

i.e.,

$$y = \frac{32}{3}x - \frac{55}{3}.$$

5. [10 points] Find $\dfrac{d}{dx}\sqrt{x}$ using the limit definition of the derivative.

Solution: Using the definition of the derivative we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h}-\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

6. [10 points] Consider the curve defined by

$$y^2 = kx + \sin(x - y)$$

where k is a constant. Use implicit differentiation to find y'.

Solution: First, implicitly differentiate to get

$$2yy' = k + \cos(x - y)(1 - y').$$

If we solve this equation for y' we get

$$y' = \frac{k + \cos(x - y)}{2y + \cos(x - y)}.$$

7. [10 points] Suppose

$$\frac{d}{dx}\big(f(2x^3+1)\big) = 4x^2.$$

Find $f'(x)$.

Solution: Note that

$$\frac{d}{dx}\big(f(2x^3+1))\big) = f'(2x^3+1)(6x^2) = 4x^2.$$

Dividing through by $6x^2$ shows that $f'(2x^3+1) = 2/3$. But every real u can be wirtten in the form $2x^3+1$; we have only to choose x as the cube root of $(u-1)/2$. It follows that f' is always equal to $2/3$; in particular, $f'(x) = 2/3$.

8. [10 points] The sweet sounds of an ice cream truck float by as Calvin and Hobbes enjoy the last day of summer riding their bikes around the neighborhood. Calvin can tell that the truck is nearby, but there seems to be no direct bike route he can take from the corner of Main and School Street to get his hands on a scrumptious Rocket Pop. At precisely 12:30, he and Hobbes decide to split up–Calvin will head due north along Main and Hobbes will ride due west down School Street. At 12:31, Calvin is 300 feet away from the intersection and he is still running north at a speed of 300 ft./min. At the same time, Hobbes is 400 feet away from the intersection and he is still hopping west at a speed of 900 ft./min. How fast are they moving apart from each other at that time?

Solution: Let $x(t)$ be Hobbes's distance from the intersection at time t and $y(t)$ Calvin's distance (both measured in feet). By the Pythagorean theorem, the distance $d(t)$ between Calvin and Hobbes satisfies

$$d(t)^2 = x(t)^2 + y(t)^2.$$

Differentiating with respect to t shows

$$2d(t)d'(t) = 2x(t)x'(t) + 2y(t)y'(t).$$

At the time t_0 corresponding to 12:31, Hobbes is 400 feet from the intersection (i.e., $x(t_0) = 400$) and Calvin is 300 feet away (so $y(t_0) = 300$); thus

$$d(t_0)^2 = 400^2 + 300^2,$$

and hence the distance $d(t_0)$ between the two is 500 feet. We are also given that $x'(t_0) = 900$ and $y'(t_0) = 300$. Substituting into the above equation shows that at t_0,

$$2 \cdot 500 \cdot d'(t_0) = 2 \cdot 400 \cdot 900 + 2 \cdot 300 \cdot 300.$$

Thus

$$d'(t_0) = \frac{2 \cdot 400 \cdot 900 + 2 \cdot 300 \cdot 300}{2 \cdot 500} = 900 \frac{\text{ft.}}{\text{min.}}$$

9. [5 points] What is

$$\lim_{x \to \pi/4} \frac{\tan(x) - \tan(\pi/4)}{x - \pi/4} \quad ?$$

Solution: We observe that this limit is actually a derivative. So if we let $f(x) = \tan(x)$ we have

$$\lim_{x \to \pi/4} \frac{\tan(x) - \tan(\pi/4)}{x - \pi/4} = f'(\pi/4) = \sec^2(\pi/4) = 2.$$

This page is for scratch work.

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