Case Study: Flood Watch

Animation: Flood Watch **To get you going on the Case Study!**

In this section, we have learned that if we are given the continuous derivative f' of a function on the interval $[a, b]$, then one version of the Fundamental Theorem of Calculus states that

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

This formula is very useful if we can find an antiderivative for f'; that is, if we have an explicit representation of f. However, in an applied setting, we may not have a formula for f and thus cannot use the fundamental theorem directly. The derivative $f'(x)$ may be given at only a finite number of points of the interval, and that may be all that is known. That and the fact that $f'(x)$ is a continuous rate of change of a function f. In these cases, we want to keep in mind that we can approximate the integral using a Riemann sum. This is the procedure that we will follow in the CSC of this section where the derivative is a non-negative function on an interval. We will use the Riemann sum

$$\sum_{i=1}^{n} f'(x_i)\Delta x_i$$

as an approximation to the area under the graph of the continuous function f'.

As usual, the purpose of a CSC is to consider a real application of calculus, with real data. In this section, we will work with our earth scientist colleagues to study problems related to river flooding [1]. Without being too specific about our objective at this time, we will state it in general terms so that we have an idea where we are going.

Objective: From data collected ten years apart, to determine if the Gorge River has an increased or decreased likelihood of flooding.

In order to make the objective more concrete, we are going to have to learn how earth scientists collect and assemble information about rivers. We will play the role of mathematicians working with geologists to provide the mathematical analyses that regional and urban planners need for the management of resources having an impact on watersheds and the like. We have to be careful, though, because we will find that mathematicians and earth scientists sometimes use the same terminology for different concepts. Because this is a CSC and we want to get practice applying mathematics to a different field, we will use the earth-science terms. But be careful to translate them into their mathematical form so that you know what calculus tools to apply.

In the next paragraph, we describe all of the terms we will be using. Often the physical units (for example, volume of water per unit time) will help with the language conversions from earth sciences to mathematics.

River Flooding: Background

The flow of water on the surface of the earth in rivers and streams has an important impact on humans. Many of these water bodies provide important resources as navigation routes and sources of fresh water for residential and industrial use. The erosion force of the streams is responsible for the formation of many of our land forms both by removal as well as deposition of material. Perhaps their most dramatic impact is flooding, when the discharge (volume of water per unit time) exceeds the normal carrying capacity of the channel, and the water spills out onto the adjacent flood plain. In an ideal world, flood plains would be reserved

[1] The CSC is adapted from Skinner, B. J. and Porter, S. C., 1962, *The Dynamic Earth, An Introduction to Physical Geology*, 2nd ed., Wiley, 570 pp.; and from Leopold, L. B., 1968, *Hydrology for Urban Land Planning*, a guidebook on the hydrologic effects of urban land use, U.S. Geological Survey Circular 554. We also owe special thanks to Professor Dick Birnie, Department of Earth Sciences, Dartmouth College, for collaborating on this work.

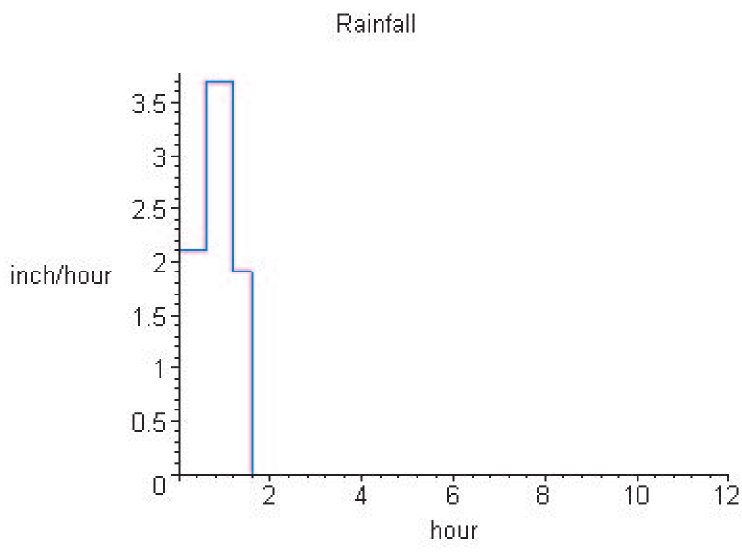
for rivers and perhaps agriculture which rejuvenates from fresh silt and natural mineral fertilizers left by the river. However, the transportation and recreational value of rivers has lured many human settlements to river banks. Over time, many of these settlements have grown into major metropolitan centers such as St. Louis and New Orleans on the Mississippi River. When a river floods, the human impact is disastrous. People who live and work along a river should not be surprised by a flood. Floods are regular and expected events; they have happened in the past and they will continue to happen in the future.

Earth Scientists make continuous observations of the discharge (volume of water per unit time) of a stream. These observations lead to an understanding of the flow characteristics of the river and its watershed (area that drains into the river) and allow predictions of future behavior. This concept repeats itself throughout Earth Science: observations (measurements) are made, an understanding (perhaps a mathematical model) is derived by analyzing (tabulating, graphing, statistically processing) the observations; and then predictions are extrapolated from the model.

The fundamental way to display the observations of stream discharge is a *hydrograph*. A hydrograph has two components: the amount of rainfall (depth per unit time) and discharge (volume of water per unit time) (see the example below). Prior to a rain storm, the stream will be flowing at some background level of discharge known as base flow. However, following a period of heavy precipitation, the rain falling in the watershed drains into the stream, and the discharge increases over time. The plot below shows the amount of discharge over a 12 hour period following an intense rainstorm lasting about 96 minutes. The discharge increases rapidly to a peak about 3 hours after the storm and then gradually drops for the next 9 hours. The discharge of a stream does not rise immediately with the onset of precipitation, rather it takes time to flow across the watershed and into the stream. The difference in time between the *centroid* of the precipitation and the centroid of the discharge (runoff) is known as the *lag time* and that depends on the size and makeup of the watershed. Neglecting absorption by the ground, the area of the watershed times the depth of the rainfall equals the volume of discharge above the baseflow. If the total volume of discharge of the stream exceeds the carrying capacity of the channel, the stream overflows its banks and floods.

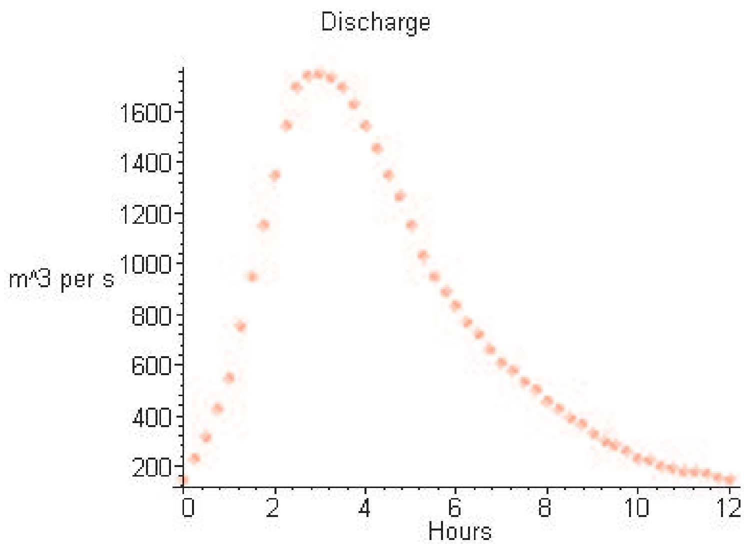
Example of a Hydrograph

A hydrograph consists of two parts: a plot of rainfall, and a plot of discharge. Here is a set of possible data giving the discharge of a specific stream caused by an intense amount of rainfall. First, the rainfall plot in inches per hour:



The rainfall is 2.1 $\frac{in}{hr}$ for the first 0.6 hours, then it is 3.7 $\frac{in}{hr}$ for the next 0.6 hours, and finally it is 1.9 $\frac{in}{hr}$ for 0.4 hours.

Next, the data representing the discharge D of a stream at specific instants of time beginning at the onset of the intense downpour of rain given above are as follows, where the discharge is measured in cubic meters per second:



Notice that although the discharge rises rather quickly, it drops slowly back to the base flow level of the stream.

To find the lag time, we need to compute the centroids of the plots of rainfall and discharge. The **centroid** of a plane region is the point (x_0, y_0) on which the region, thought of as a thin plate, would balance horizontally. To find the point, we use the formulas for the coordinates.

Theorem: Suppose the region of area A is defined as lying between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ and $a \leq x \leq b$. Then its centroid is located at the point (x_0, y_0) given by

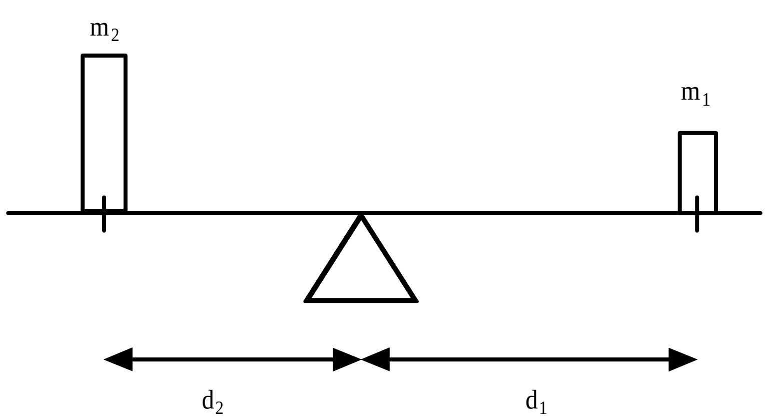
$$x_0 = \frac{1}{A}\int_a^b x(f(x) - g(x))\,dx$$

$$y_0 = \frac{1}{A}\int_a^b \frac{f(x)^2 - g(x)^2}{2}\,dx$$

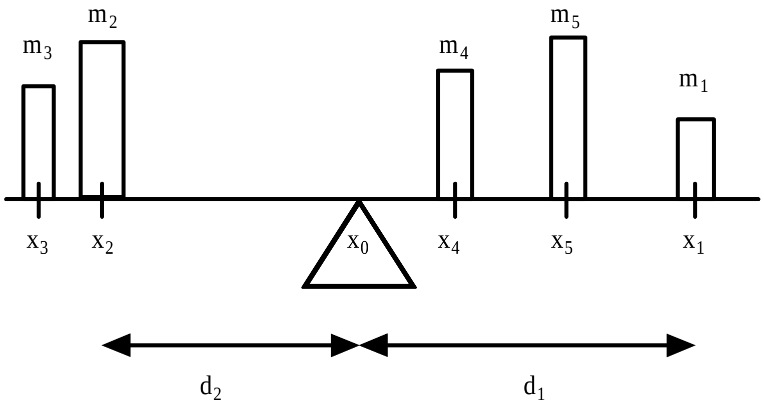
Because the proof is instructive for our discussion of the hydrograph, we will justify this formula before proceeding further with the rainfall data.

Derivation of Formulas for the Centroid

Suppose two masses m_1 and m_2 are placed on a see-saw, at distances d_1 and d_2, respectively, from the pivot-point. Then how far do we have to move one mass relative to the other so that the see-saw will balance?



This is an old problem that has a very simple solution: the see-saw will balance when $m_1 d_1 = m_2 d_2$. For example, if m_2 is large relative to m_1, then d_2 must be small relative to d_1. We all have experienced the effects of this equation: the larger person has to sit closer to the pivot point to balance the smaller person sitting at the opposite end of the see-saw.



Suppose now that we consider n masses $m_1, m_2, m_3, \ldots, m_n$ located on a see-saw at the points $x_1, x_2, x_3, \ldots, x_n$, and suppose further that the pivot point is located at x_0. Then generalizing from the case of two masses, it turns out that balance is achieved when

$$\sum_{k=1}^{n} m_k(x_k - x_0) = 0$$

Note that if $x_k > x_0$, then $x_k - x_0$ is positive and is the distance of m_k from the pivot-point, while if $x_k < x_0$, then $x_k - x_0$ is negative and is the negative of the distance of m_k from the pivot-point. So, this equation is indeed a generalization of the one we started with involving only two masses. If we solve for x_0 in this equation, then we get that

$$x_0 \sum_{k=1}^{n} m_k = \sum_{k=1}^{n} m_k x_k$$

Calling $\sum_{k=1}^{n} m_k$ the *total mass*, we then have that

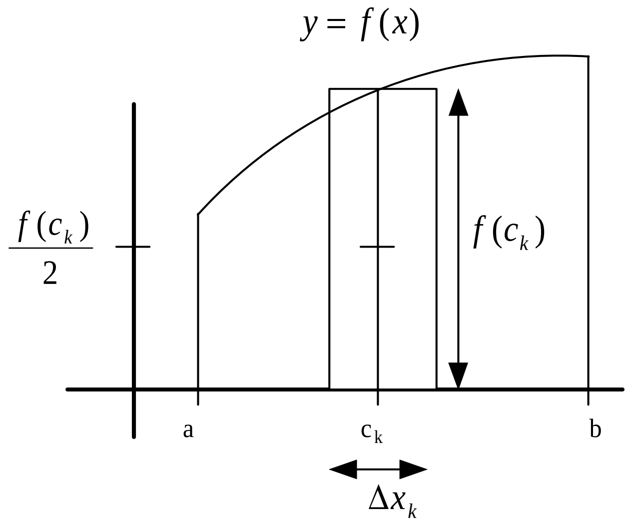
$$x_0 = \frac{\sum_{k=1}^{n} m_k x_k}{\text{total mass}}$$

By analogy with masses, if the rectangles pictured above are simply two-dimensional regions, we can write a similar formula for the coordinate of the pivot-point using areas:

$$x_0 = \frac{\sum_{k=1}^{n} x_k A_k}{\text{total area}}$$

where A_k is the area of the k^{th} rectangle. The product $x_k A_k$ is called the *moment* of the k^{th} rectangle.

Suppose now that we start with a function $y = f(x)$ defined on an interval $[a, b]$, and as usual we partition the interval into n subintervals with the points $x_0 < x_1 < x_2 < \cdots < x_n$. Let c_k be the midpoint of the k^{th} subinterval. Then $c_k A_k = c_k f(c_k)\Delta x_k$ is the moment of the k^{th} rectangle with respect to the y-axis.



We next recognize that the sum of the moments of the n rectangles is a Riemann sum whose limit is the definite integral (assumed to exist) of the function over the interval $[a, b]$:

$$\lim_{n\to\infty}\sum_{k=1}^{n} c_k f(c_k)\Delta x_k = \int_a^b x f(x)\, dx$$

In a similar manner, we can compute the moment of the k^{th} rectangle with respect to the x-axis. Because the y-coordinate of the center of the rectangle is $\frac{f(c_k)}{2}$, the moment with respect to the x-axis is $\frac{f(c_k)}{2} f(c_k)\Delta x_k$, that is, the y-coordinate of the center times the area of the rectangle. Once again, we recognize the sum of these moments as a Riemann sum, whose limit is a definite integral:

$$\lim_{n\to\infty}\sum_{k=1}^{n} \frac{f(c_k)}{2} f(c_k)\Delta x_k = \frac{1}{2}\int_a^b f(x)^2\, dx$$

Finally, the point (x_0, y_0) whose coordinates satisfy

$$Ax_0 = \int_a^b x f(x)\, dx$$

$$Ay_0 = \frac{1}{2}\int_a^b f(x)^2\, dx$$

where A is the area under the graph of f, is called the *centroid* of the region. It is the balance point of the area of the region.

Back to Example of a Hydrograph

We had gotten to the point in our discussion of the hydrograph where we were about to compute the centroids of the rainfall and discharge data.

The rainfall plot is composed of three rectangles. We refer to the proof above of the centroid formula to find the x-coordinate of the centroid. The proof shows that we can calculate it as the sum of the midpoint of a rectangle times its area (the height of the rectangle times the width of the rectangle), all divided by the area of the rainfall plot.

The area of the rainfall data is

$$A = .6(2.1) + .6(3.7) + .4(1.9) = 4.24$$

Hence, we find that the coordinates of the centroid of the rainfall data are:

$$x_0 = \frac{1}{A}\left(.3(2.1)(.6) + .9(3.7)(.6) + 1.4(1.9)(.4)\right) \approx .81132$$

$$y_0 = \frac{1}{2A}\left((2.1^2)(.6) + (3.7^2)(.6) + (1.9^2)(.4)\right) \approx 1.45094$$

Applet: Flood Watch **Try it!**

To find the centroid of the discharge data, which is given in a convenient computational form such as an applet or a Maple worksheet, we use a Riemann sum approximation to calculate the integrals in question. First, letting y_{min} (= 150 in this example) be the base level of the flow, the (approximate) area A under the discharge graph is obtained as the sum

$$A = 3600 \sum_{k=1}^{n} (f(c_k) - y_{\text{min}})\Delta x_k \approx 7418$$

where we have shown explicitly the multiplier 3600 $\frac{sec}{hr}$ that is needed to make consistent the units on the vertical and horizontal axes.

Then, the x and y coordinates of the centroid of the discharge data due to rainfall are:

$$x_0 = \frac{1}{A} \sum_{k=1}^{n} c_k(f(c_k) - y_{\text{min}})\Delta x_k \approx 4.294$$

$$y_0 = \frac{1}{2A} \sum_{k=1}^{n} (f(c_k)^2 - y_{\text{min}}^2)\Delta x_k \approx 691.232$$

Note that x_0 is in hours, and y_0 in $\frac{m^3}{sec}$. The lag time is the difference of the x-values of the centroids of the discharge and rainfall data, or approximately 3.4827 hours. Moreover, if we wanted a measure of the intensity of the storm, we could compute the difference in the y-values of the centroids.

To see where the above formulas for x_0 and y_0 come from, note that the area of one rectangle for computing the x coordinate x_0 and the y coordinate y_0 of the centroid is

$$\left(f(c_k) - y_{\text{min}}\right)\Delta x_k.$$

The distance of the rectangle's centroid from the y-axis is c_k, and its distance from the x-axis is

$$\frac{1}{2}\left(f(c_k) + y_{\text{min}}\right).$$

Thus, each term in the sum for computing x_0 is $\frac{1}{A} c_k \left(f(c_k) - y_{\text{min}}\right)\Delta x_k$, and each term in the sum for computing y_0 is

$$\frac{1}{2A}\left(f(c_k) + y_{\text{min}}\right)\left(f(c_k) - y_{\text{min}}\right)\Delta x_k = \frac{1}{2A}\left(f(c_k)^2 - y_{\text{min}}^2\right)\Delta x_k$$

and both formulas are obtained as above by summing the respective terms.

The CSC: Flood Watch

Now that we have considered an example of a hydrograph, and discussed the relevant mathematics, we are ready to state the objective, setup, and thinking and exploring issues for the CSC.

Objective: We are given two hydrographs for the Gorge River that have been recorded ten years apart. The overall objective is to analyze, compare, and interpret the two hydrographs, thereby reaching a conclusion about the increased or decreased likelihood of the Gorge River flooding.

Setup: We will be plotting the rainfall and discharge data, and assuming that the discharge represents a sample of measurements drawn from a continuous function. We will be using Riemann sums to calculate approximations to the area under the curve, and to the centroid of the discharge data. We will keep track of the various units of measurement to be certain that we have a consistent set.

Thinking and Exploring: We will be studying each hydrograph, and then comparing our findings for the two. The rainfall is measured in units of inches per hour, the discharge in units of cubic meters per second, and the horizontal time line in hours.

Some of the questions we will consider under Thinking and Exploring are: What is the average base flow? What is the time at which the discharge is maximum? What is the maximum discharge? What is the discharge due to base flow over the 12 hour observation time? What is the discharge due to the rain storm over the 12 hour observation time? What is the total discharge over the 12 hour observation time? About how long did the rain storm last? What is the center of mass of the precipitation? What is the center of mass of the discharge? What is the lag time? What is the area of the watershed?

Applet: Flood Watch **Try it!**

As ususal, the last step in a CSC is to write a summary of the investigations.

Interpretation and Summary: After studying the two hydrographs of the Gorge River, and thinking about the mathematical tools, it is time to interpret and summarize the mathematical results in terms of the original objective.

Pretend that your synopsis is going to appear in the next issue of a magazine such as *Scientific American*. Include enough details so that a reader would learn what the major issues of the report are, and how you went about addressing them. What will you want to tell readers about your success with regard to the original stated objective of the investigation? Be sure to write in complete sentences using correct rules of standard English grammar.

To get you started, here are two questions that definitely need to be answered:

1. What changes in the watershed might have occurred in the 10 years between the hydrographs to account for the different pattern of the more recent hydrograph relative to the earlier one?

2. How might these changes influence the likelihood of the Gorge River flooding?

What advice do you have for public policy makers? Make the report interesting, compelling. Ask yourself: Would a policy maker be able to understand it, and feel compelled to follow its recommendations?

Exercises: Problems **Check what you have learned!**

Videos: Tutorial Solutions **See problems worked out!**