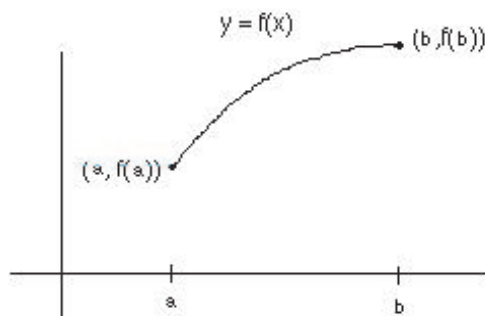


Arc Length

©2002 Donald Kreider and Dwight Lahr

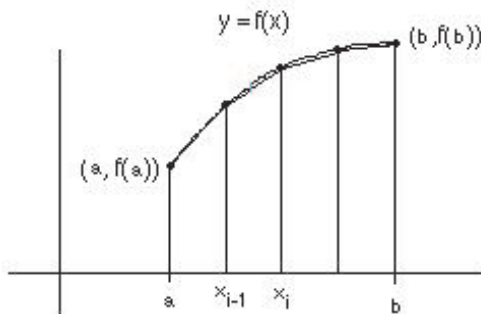
We have seen that in applications integrals are very powerful modeling tools through the use of Riemann Sums. As another illustration of the Riemann Sum modeling method, consider the problem of computing the length of a curve in the plane. To be explicit, we will assume that $y = f(x)$ is a continuous function defined on the interval $[a, b]$ and that $f'(x)$ exists at every point of the interval. Then our task is to determine the length of the graph of f from the point $(a, f(a))$ to the point $(b, f(b))$.



If the graph were a straight line, we could use the formula—that comes from the Pythagorean theorem—for the distance between two points to find the length of the line connecting them. If the graph is not a straight line, then it makes sense to do what we probably would have done in grade school, namely, use a finite number of straight line segments to approximate the length of the curve. We then can use the Riemann Sum approach to add the lengths, and pass to the limit as all lengths go to zero. This procedure should allow us to compute the length exactly. Let's outline the specific steps.

Summary of the Riemann Sum Method for Arc Length: Here are the steps in the modeling process of using Riemann Sums to find the arc length of a curve in the plane:

1. Divide the interval $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$. Call the points of the subdivision $a = x_0 \leq x_1 \leq x_2 \leq x_3 \leq \dots \leq x_{n-1} \leq x_n = b$, where $x_i = a + i\Delta x$ for each i .
2. On each subinterval $[x_{i-1}, x_i]$ connect the points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ on the graph of f with straight lines.



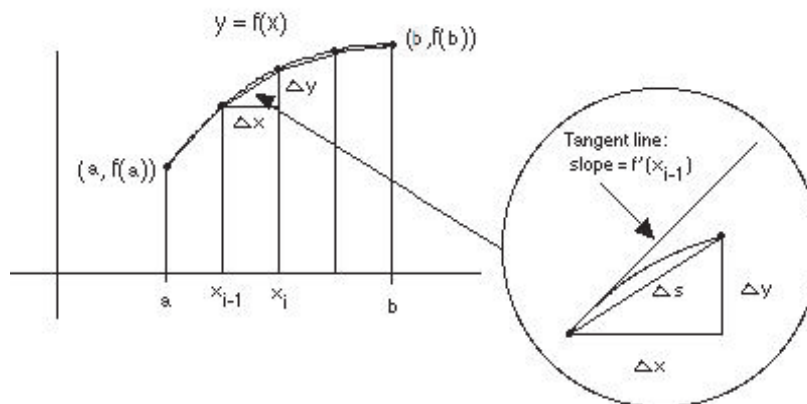
Now, let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y = f(x_i) - f(x_{i-1})$. Then the length of the straight-line segment connecting the two points is:

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\Delta s = \sqrt{(\Delta x)^2 \left(1 + \frac{(\Delta y)^2}{(\Delta x)^2}\right)} = \Delta x \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}}$$

But for small Δx , the slope $\frac{\Delta y}{\Delta x}$ of the line is approximately equal to the slope of the tangent line at the left-hand point, which equals $f'(x_{i-1})$.



Thus, we can replace Δs by the approximate value $\Delta s \approx \Delta x \sqrt{1 + [f'(x_{i-1})]^2}$. We will use this approximate value for the length of the straight-line segment on each subinterval.

- The sum of the approximate lengths of these line segments provides an approximation to the length of the curve:

$$\sum_{i=1}^n \sqrt{1 + [f'(x_{i-1})]^2} \Delta x$$

- Taking the limit as $\Delta x \rightarrow 0$, the above approximation approaches the length of the curve. Moreover, we recognize it as a limit of left-hand Riemann Sums. Thus, the limit is the definite integral

$$L = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n \sqrt{1 + [f'(x_{i-1})]^2} \Delta x = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

- Now, use the integral formula to compute the length L of the graph of f between $x = a$ and $x = b$.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

The formula is called the *arc length* formula.

In many cases the arc length formula leads to an integrand for which we do not know an antiderivative and hence cannot apply the Fundamental Theorem of Calculus. In those situations, we still can find an accurate expression for the length of the graph by evaluating the definite integral using the Trapezoid Rule (or some other numerical method). In this age of calculators and computers, the Trapezoid Rule presents

no practical obstacle, and hence the arc length formula is all we need to calculate the length of the graph of a function over an interval.

Example 1: To find the length of the arc $y = x^{3/2}$, from $x = 0$ to $x = 1$, we use the Arc Length formula:

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left[\frac{3}{2}x^{1/2}\right]^2} dx \\ &= \int_0^1 \sqrt{1 + \left[\frac{9}{4}x\right]} dx \end{aligned}$$

This is a substitution integral with $u = 1 + \frac{9}{4}x$; then $du = (9/4)dx$. Also, changing the limits of integration, $u(0) = 1$ and $u(1) = 13/4$. So,

$$\begin{aligned} L &= \frac{4}{9} \int_1^{13/4} \sqrt{u} du \\ &= \frac{8}{27} u^{3/2} \Big|_1^{13/4} \\ &= \frac{13^{3/2} - 8}{27} \end{aligned}$$

Example 2: Find the length of the curve $y = x^4 + \frac{1}{32x^2}$ from $x = 1$ to $x = 2$. We calculate y' which we need for the arc length formula:

$$y' = 4x^3 - \frac{2}{32x^3} = 4x^3 - \frac{1}{16x^3}$$

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + \left[4x^3 - \frac{1}{16x^3}\right]^2} dx \\ &= \int_1^2 \sqrt{1 + 16x^6 - \frac{8}{16} + \frac{1}{256x^6}} dx \\ &= \int_1^2 \sqrt{\frac{8}{16} + 16x^6 + \frac{1}{256x^6}} dx \\ &= \int_1^2 \sqrt{\left(4x^3 + \frac{1}{16x^3}\right)^2} dx \\ &= \int_1^2 \left(4x^3 + \frac{1}{16x^3}\right) dx \\ &= \left(x^4 - \frac{1}{32x^2}\right) \Big|_1^2 \\ &= 15 + \frac{3}{128} \end{aligned}$$

Applet: [Numerical Integration](#) Try it!

Exercises: [Problems](#) Check what you have learned!

Videos: [Tutorial Solutions](#) See problems worked out!