

## Volumes of Solids of Revolution

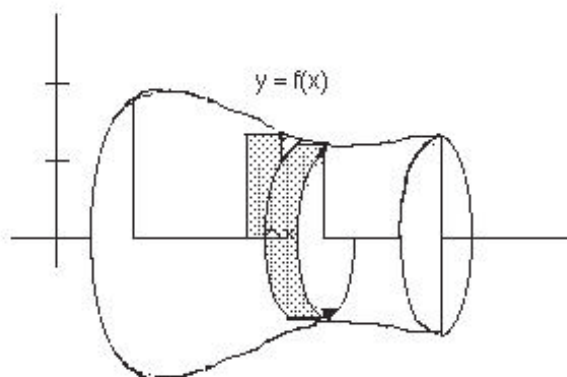
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Integrals find application in many modeling situations involving continuous variables such as area or volume. They allow us to model physical entities that can be described through a process of adding up, or accumulating, smaller infinitesimal parts. In what follows, we will illustrate by discussing the very powerful *Riemann Sum* approach to modeling with integrals.

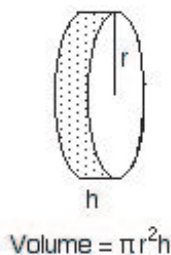
Riemann Sum modeling is based on describing real-world phenomena through continuous functions on closed intervals. In particular, Riemann Sums provide a method of analysis that proceeds by dividing an interval into finitely many small subintervals; developing on each individual subinterval a function-related formula that works there; summing these individual contributions; and passing to the limit as the length of the largest subinterval goes to zero. At the end of this process, the resulting integral computes, or even defines, the quantity that was desired at the outset.

We have already used this approach to describe the area under the graph of a continuous function and above a closed interval. Now we are going to use it to find the volume of a so-called *solid of revolution*.

We begin with a plane region  $R$  bounded by the non-negative function  $y = f(x)$ ,  $y = 0$ ,  $x = a$ , and  $x = b$ . We then rotate this region about the  $x$ -axis.



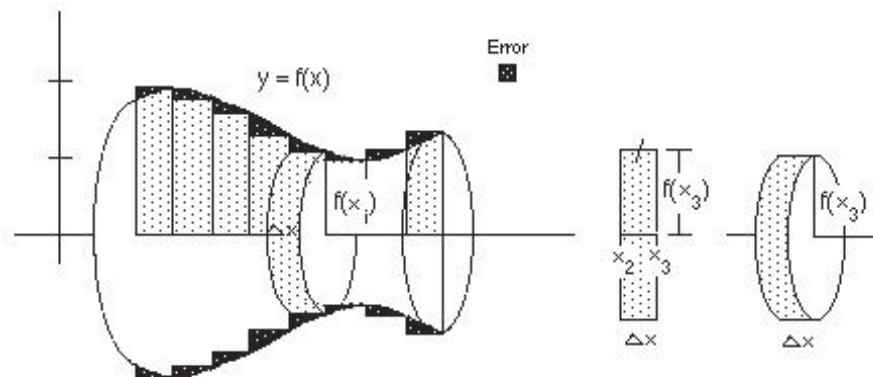
The resulting three-dimensional solid is called a *solid of revolution*. Note that its cross-sectional area in a plane perpendicular to the  $x$ -axis at  $x$  is a circular disk of radius  $f(x)$ . Also note that when we rotate a rectangular area element, we get a circular disk. We are going to use the volumes of these disks to approximate the volume of the solid of revolution. A right circular disk of radius  $r$  and width  $h$  has volume  $\pi r^2 h$  (the area of the circular base times the width).



**Summary of the Riemann Sum Volume of Revolution Method:** In light of the description above of the Riemann Sum method to compute volumes of solids of revolution, we can summarize the general procedure that we will apply in different situations. These steps capture the essence of the modeling approach using Riemann Sums to find these volumes.

Begin with a continuous non-negative function  $f$  on a closed interval  $[a, b]$ . Revolve the graph of  $f$  around

the  $x$ -axis to obtain a so-called *solid of revolution*. The problem is to compute its volume. To do this, proceed as follows:



1. Divide the interval  $[a, b]$  into  $n$  subintervals of equal length  $\Delta x = (b - a)/n$ . Call the points of the subdivision  $a = x_0 \leq x_1 \leq x_2 \leq x_3 \leq \dots \leq x_{n-1} \leq x_n = b$ , where  $x_i = a + i\Delta x$  for each  $i$ .
2. Erect on each subinterval  $[x_{i-1}, x_i]$  a rectangle of height  $f(x_i)$ ; then revolve this rectangle about the  $x$ -axis to generate a circular disk of radius  $f(x_i)$  and width  $\Delta x$ . The volume of the disk is thus  $\pi f(x_i)^2 \Delta x$ .
3. The sum of the volumes of these disks provides an approximation to the volume of the solid of revolution:

$$\sum_{i=1}^n \pi f(x_i)^2 \Delta x$$

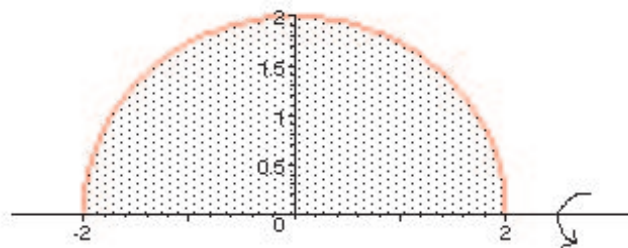
4. Taking the limit as  $\Delta x \rightarrow 0$ , the above approximation approaches the volume  $V$  of the solid of revolution. Moreover, we also recognize it as a limit of Riemann Sums that converge to the definite integral:

$$V = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n \pi f(x_i)^2 \Delta x = \int_a^b \pi f(x)^2 dx$$

5. Finally we use the integral formula to compute the volume  $V$  of the solid of revolution.

$$V = \pi \int_a^b f(x)^2 dx$$

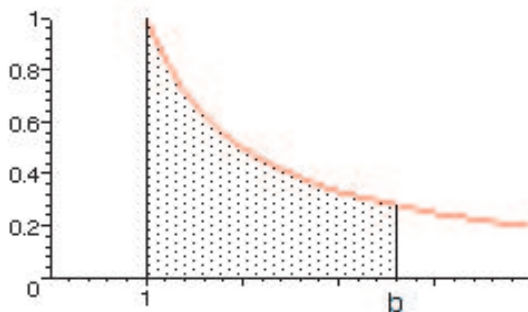
**Example 1:** Find the volume of a ball of radius 2. To solve this problem, we begin with a plane region which, when revolved about the  $x$ -axis, will generate the ball. To this end, we let  $y = \sqrt{4 - x^2}$  define the upper boundary of a semicircle on the interval  $[-2, 2]$ .



Then using the formula from above, the volume of the ball is

$$V = 2\pi \int_0^2 (\sqrt{4 - x^2})^2 dx = 2\pi \int_0^2 (4 - x^2) dx = 2\pi \frac{16}{3}$$

**Example 2:** Revolve the region bounded on top by  $y = 1/x$  and above the interval  $[1, b]$ , where  $b > 1$ , about the  $x$ -axis as pictured.



Then the volume is

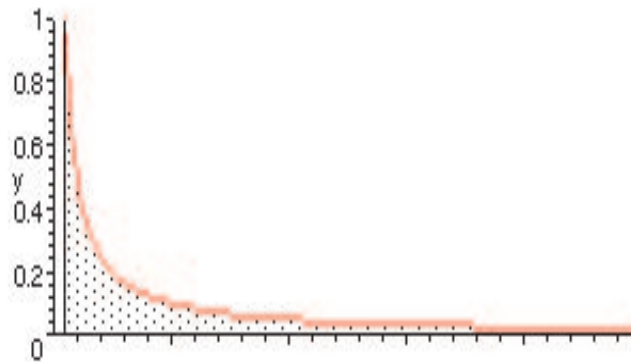
$$V = \pi \int_1^b x^{-2} dx = \pi \left(1 - \frac{1}{b}\right)$$

Thus, as  $b \rightarrow \infty$ , the volume of the 3-D pointed horn approaches the value  $\pi$ .

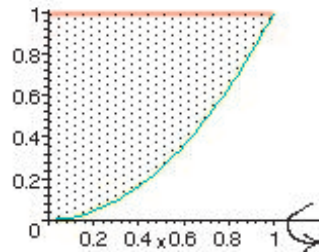
On the other hand, if we compute the area of the plane region above the interval  $[1, b]$  and under the graph of  $y = 1/x$ , we get:

$$A = \int_1^b \frac{1}{x} dx = \ln b$$

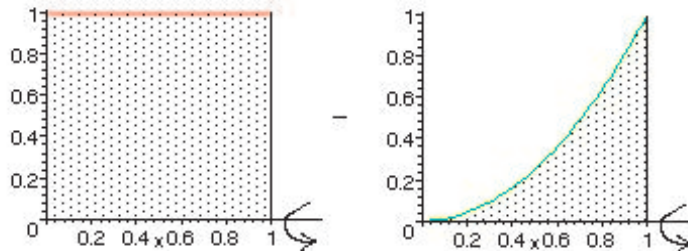
Thus, the area approaches  $\infty$  as  $b \rightarrow \infty$ . So, the solid of revolution has finite volume whereas the planar figure that generated it has infinite area. Curious, eh?



**Example 3:** Revolve the region bounded by  $y = x^2$  and  $y = 1$  and  $x \geq 0$  about the  $x$ -axis and compute the volume of the resulting solid. This solid will hold water if we turn it on its side.



We think of generating it by revolving the two plane regions shown and subtracting the 3-D results:

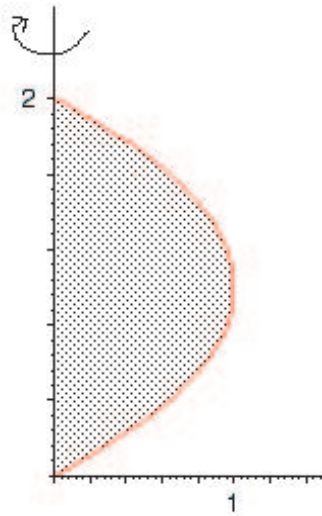


The formula we have developed for volume applies to each of these two situations. Thus, the volume of the solid we seek is:

$$V = \pi \int_0^1 1^2 dx - \pi \int_0^1 (x^2)^2 dx = \pi \int_0^1 (1 - x^4) dx = \frac{4}{5}\pi$$

**Horizontal Rectangular Elements:** We can also revolve a region about the  $y$ -axis using horizontal rectangular elements.

**Example 4:** Revolve the region bounded by  $x = 0$  and  $x = 2y - y^2$  about the  $y$ -axis and compute the volume of the resulting solid.



The volume is

$$V = \pi \int_0^2 (2y - y^2)^2 dy = \pi \int_0^2 (4y^2 - 4y^3 + y^4) dy = \frac{16}{15}\pi$$

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