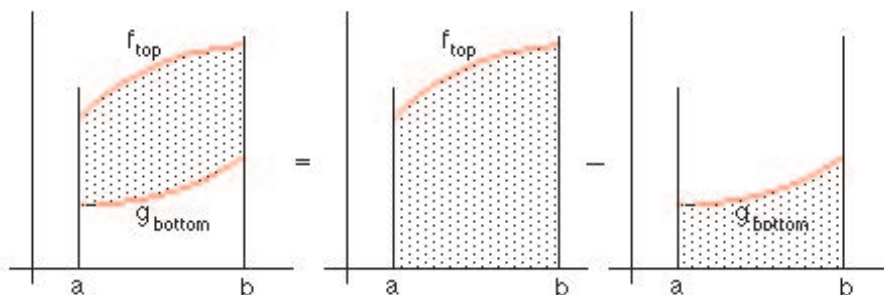


Areas Between Curves

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We know that if f is a continuous nonnegative function on the interval $[a, b]$, then $\int_a^b f(x) dx$ is the area under the graph of f and above the interval. We are going to extend this notion a bit by considering how to find the area between two functions. To be specific, suppose we are given two continuous functions, f_{top} and g_{bottom} defined on the interval $[a, b]$, with $g_{bottom}(x) \leq f_{top}(x)$ for all x in the interval. How do we find the area bounded by the two functions over that interval?



We have used the notation f_{top} and g_{bottom} for obvious reasons. However, we want to caution you that all of the subsequent analysis really does assume that f lies above or is equal to g at every point throughout the interval. So, you want to be sure in problem-solving that you have verified that this is the case before using the formula that we will develop next.

The area of the region between the two curves and above the interval $[a, b]$ equals the area of the region under the graph of f_{top} on that interval minus the area of the region under the graph of g_{bottom} on the same interval. Thus, the area of the region between the two curves equals

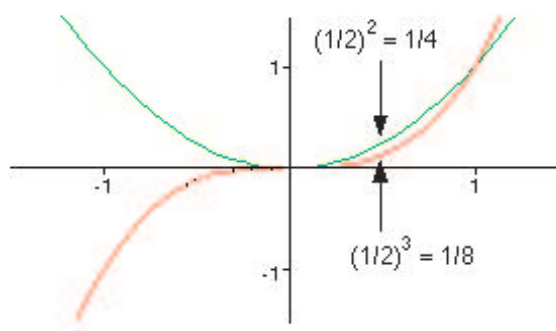
$$\int_a^b f_{top}(x) dx - \int_a^b g_{bottom}(x) dx = \int_a^b (f_{top}(x) - g_{bottom}(x)) dx$$

The last integral is normally the form in which we express the area between the two curves. But remember that to apply the formula, we have to know which curve is on top and which is on bottom, and we have to be certain that that relationship is maintained throughout the interval. Note though that the vertical placement of the curves does not matter. For example, both of them could lie below the x -axis, or one above and one below, and the formula would still hold.

Example 1: Find the area of the region between the graphs of $y = x^2$ and $y = x^3$ for $0 \leq x \leq 1$. Solving for the point(s) of intersection, we find that the curves intersect at $x = 0, 1$:

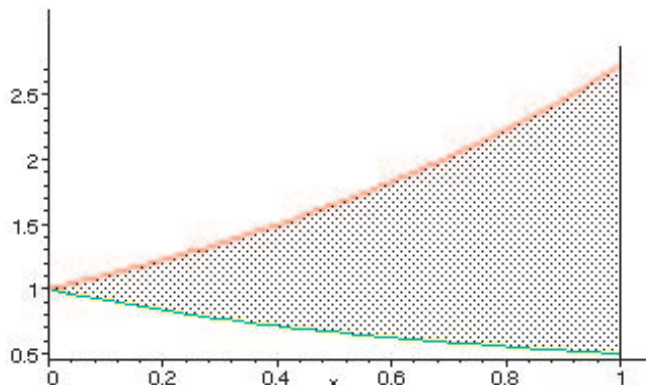
$$\begin{aligned} x^2 &= x^3 \\ x^2(x - 1) &= 0 \end{aligned}$$

implies $x = 0, x = 1$. Here is what a quick sketch by hand might look like:



To determine which curve is on top, we plug in a convenient value of x to find that $y = x^2$ is on top throughout the interval $[0, 1]$. Thus, we can use the formula above: The area of the region between the two curves equals $\int_0^1 (x^2 - x^3) dx = (x^3/3 - x^4/4)|_0^1 = 1/3 - 1/4 = 1/12$.

Example 2: Find the area of the region between $y = e^x$ and $y = \frac{1}{1+x}$ on the interval $[0, 1]$. The graph is shown.



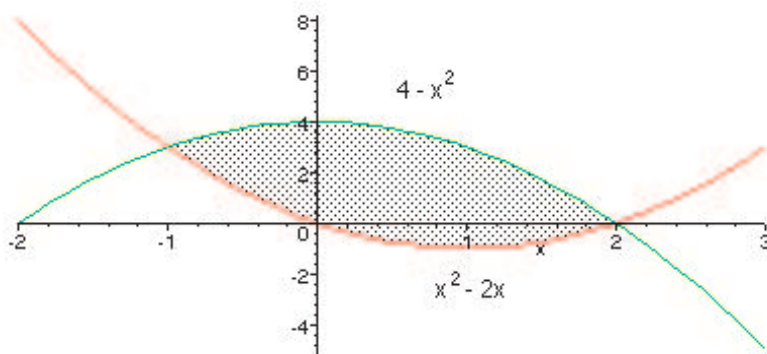
Because the curves intersect at $x = 0$, and $y = e^x$ is increasing while $y = 1/(1+x)$ is decreasing, we know that e^x is on top. Thus, the area of the region between the curves equals

$$\begin{aligned} \int_0^1 \left(e^x - \frac{1}{1+x} \right) dx &= (e^x - \ln|1+x|)|_0^1 \\ &= e - \ln 2 - e^0 + \ln 1 \\ &= e - \ln 2 - 1 \end{aligned}$$

Example 3: Find the area of the region bounded by $y = x^2 - 2x$ and $y = 4 - x^2$. To solve this problem, we need a sketch so that we can determine which function is on top over which intervals. We will begin by determining the points of intersection.

$$\begin{aligned} x^2 - 2x &= 4 - x^2 \\ 2x^2 - 2x - 4 &= 0 \\ 2(x^2 - x - 2) &= 0 \\ 2(x - 2)(x + 1) &= 0 \end{aligned}$$

So, $x = 2$ or $x = -1$. Both functions are quadratic polynomials, so their graphs are parabolas, one opening up and the other down. We should recognize from the sign on x^2 which curve is on top, or we can test a value of x to find out.



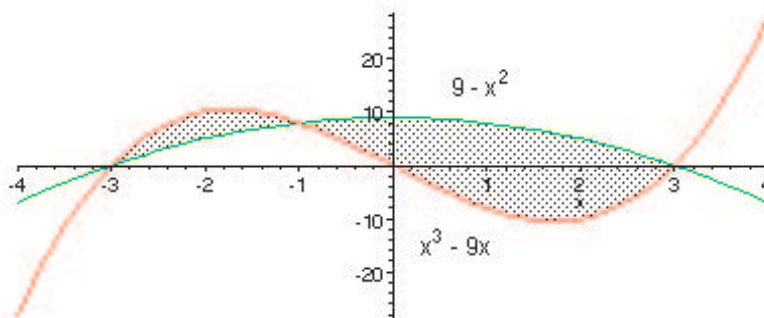
Thus, the area of the region can be gotten by applying our integral formula to obtain:

$$\begin{aligned} \int_{-1}^2 (4 - x^2 - (x^2 - 2x)) dx &= \left(4x - \frac{2x^3}{3} + x^2\right) \Big|_{-1}^2 \\ &= 8 - \frac{16}{3} + 4 - \left(-4 + \frac{2}{3} + 1\right) \\ &= 9 \end{aligned}$$

Example 4: Find the area of the region bounded by the two curves $y = x^3 - 9x$ and $y = 9 - x^2$. [Hint: You probably will need to know that $x + 1$ is a factor of $x^3 + x^2 - 9x - 9$.] Let's find the points of intersection:

$$\begin{aligned} x^3 - 9x &= 9 - x^2 \\ x^3 + x^2 - 9x - 9 &= 0 \\ (x + 1)(x^2 - 9) &= 0 \\ (x + 1)(x - 3)(x + 3) &= 0 \end{aligned}$$

Note that to obtain the next to the last equation above, we used the hint and divided $x^3 + x^2 - 9x - 9$ by $x + 1$. A sketch of the graphs follows:

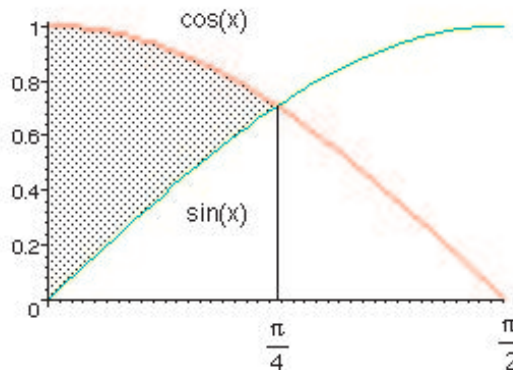


Thus, the area can be gotten by applying our integral formula twice—once to the interval $[-3, -1]$ where $x^3 - 9x$ is on top, and once to the interval $[-1, 3]$ where $9 - x^2$ is on top— and adding the results together:

$$\int_{-3}^{-1} (x^3 - 9x - (9 - x^2)) dx + \int_{-1}^3 (9 - x^2 - (x^3 - 9x)) dx$$

Routine calculation gives the answer $\frac{148}{3}$. Try it for yourself and verify this result.

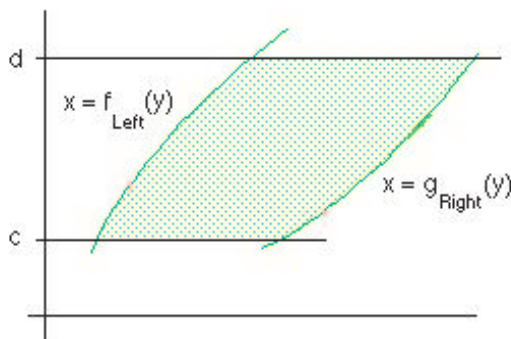
Example 5: Find the area between $\sin x$ and $\cos x$ on $[0, \pi/4]$. Here is a sketch:



So, the area is

$$\begin{aligned}
 \int_0^{\pi/4} (\cos x - \sin x) dx &= (\sin x + \cos x)|_0^{\pi/4} \\
 &= \sin(\pi/4) + \cos(\pi/4) - (\sin 0 + \cos 0) \\
 &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (0 + 1) \\
 &= \sqrt{2} - 1
 \end{aligned}$$

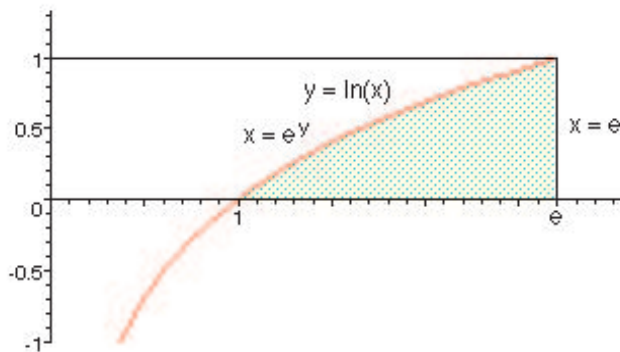
Functions of y : Thus far, we have only considered functions of x . We could just as well consider two functions of y , say, $x = f_{Left}(y)$ and $x = g_{Right}(y)$ defined on the interval $[c, d]$ on the y -axis as in the sketch below:



Then the area between the graphs can be found by subdividing the interval $[c, d]$ on the y -axis, and using horizontal rectangular area elements. In that case, we get that the area between the two curves is the definite integral

$$\int_c^d (g_{Right}(y) - f_{Left}(y)) dy$$

Example 6: Find the area under the graph of $y = \ln x$ and above the interval $[1, e]$ on the x -axis.



We know that integrating with respect to x yields the definite integral $\int_1^e \ln x dx$. However, suppose we do not know (or remember, or want to investigate) an antiderivative for $\ln x$. Then we can try to solve this problem by integrating instead with respect to y . The functions are $x = e$ on the right, $x = e^y$ on the left, over the interval from $y = 0$ to $y = 1$:

$$\int_0^1 (e - e^y) dy = (ey - e^y)|_0^1 = e - e + 1 = 1$$

Thus, our problem is solved. Alternatively, we could go back to where we began. It turns out that by application of the Integration by Parts formula to the integral $\int \ln x \, dx$, $x \ln x - x$ is an antiderivative of $\ln x$. (Just calculate the derivative to verify this result.) Thus, we can evaluate the original integral with respect to x :

$$\int_1^e \ln x \, dx = (x \ln x - x)|_1^e = e - e - (0 - 1) = 1$$

As we should, we get the same answer for the area of the region integrating with respect to either x or y .

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