Trapezoid Rule and Simpson's Rule
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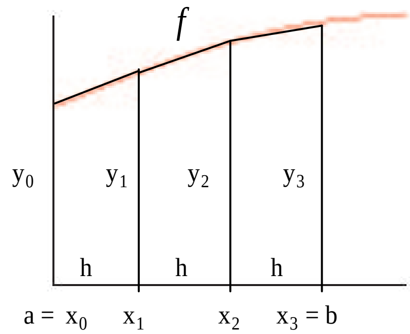
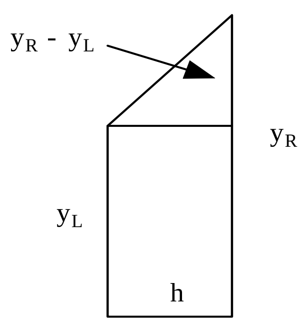
Trapezoid Rule

Many applications of calculus involve definite integrals. If we can find an antiderivative for the integrand, then we can evaluate the integral fairly easily. When we cannot, we turn to numerical methods. The numerical method we will discuss here is called the *Trapezoid Rule*. Although we often can carry out the calculations by hand, the method is most effective with the use of a computer or programmable calculator. But at the moment let's not concern ourselves with these details. We will describe the method first, and then consider ways to implement it.



The general idea is to use trapezoids instead of rectangles to approximate the area under the graph of a function. A trapezoid looks like a rectangle except that it has a slanted line for a top. Working on the interval $[a, b]$, we subdivide it into n subintervals of equal width $h = (b - a)/n$. This gives rise to the partition $a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$, where for each j, $x_j = a + jh$, $0 \leq j \leq n$. Moreover, we let $y_j = f(x_j), 0 \leq j \leq n$. That is, the vertical edges go from the x-axis to the graph of f. Consult the sketch above where we have shown a finite number of subintervals.

If we are going to use trapezoids instead of rectangles as our basic area elements, then we have to have a formula for the area of a trapezoid.



With reference to the sketch above, the area of a trapezoid consists of the area of the rectangle plus the area of the triangle, or $hy_L + (h/2)(y_R - y_L) = h(y_L + y_R)/2$. So, the area is h times the average of the lengths of the two vertical edges.

Now, we return to the original problem of finding the definite integral of a function f defined on the interval $[a, b]$. We define the *Trapezoid Rule* as follows.

Definition: The n-subinterval trapezoid approximation to $\int_a^b f(x)\,dx$ is given by

$$\begin{aligned} T_n &= \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n\right) \\ &= \frac{h}{2}\left(y_0 + y_n + 2\sum_{j=1}^{n-1} y_j\right) \end{aligned}$$

To see where the formula comes from, let's carry out the process of adding the areas of the trapezoids. Refer to the original sketch, and use the formula we derived for the area of a trapezoid. Note that when we add the areas of the trapezoids starting on the left, the area of the first , second, and third are:

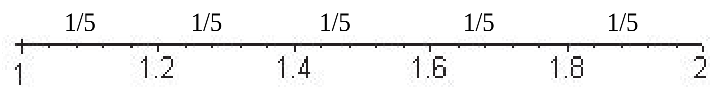
$$\frac{h}{2}(y_0 + y_1)$$

$$\frac{h}{2}(y_1 + y_2)$$

$$\frac{h}{2}(y_2 + y_3)$$

So, y_0 and y_3, the first and the last, each appear once; and all the other y_j's appear exactly twice. We can see from this example that there will be a similar pattern no matter the number of trapezoids: The first and the last vertical edge appears once, and all other vertical edges appear two times when we sum the areas of the trapezoids. This is exactly what the Trapezoid Rule entails in the formula above.

Example 1: Find T_5 for $\int_1^2 \frac{1}{x}\,dx$. We can readily determine that f(x) = 1/x, $h = 1/5$ (so $h/2 = 1/10$), and $x_j = 1 + j/5, 0 \le j \le 5$.



So,

$$T_5 = \frac{1}{10}\left(1 + \frac{1}{2} + 2\left(\frac{5}{6} + \frac{5}{7} + \frac{5}{8} + \frac{5}{9}\right)\right) \approx .0696$$

Example 2: Find T_5 for $\int_0^1 \sqrt{1-x^2}\,dx$. That is, we are going to approximate one-quarter of the area of a circle of radius 1. The exact answer is $\pi/4$, or approximately .7853981635. Note that $h = 1/5$, $y_0 = 1$ and $y_5 = 0$. Thus,

$$T_5 = \frac{1}{10}\left(1 + 2\sum_{j=1}^{4}\sqrt{1 - \frac{j^2}{25}}\right)$$

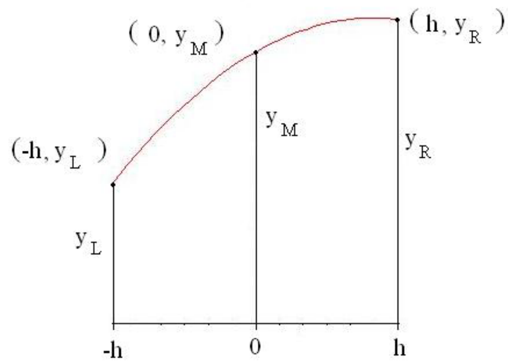
or about .7592622072.

Simpson's Rule

Another technique for approximating the value of a definite integral is called *Simpson's Rule.* Whereas the main advantage of the Trapezoid rule is its rather easy conceptualization and derivation, Simpson's rule approximations usually achieve a given level of accuracy faster. Moreover, the derivation of Simpson's rule is only marginally more difficult. Both rules are examples of what we refer to as *numerical methods*.

In the Trapezoid rule method, we start with rectangular area-elements and replace their horizontal-line tops with slanted lines. The area-elements used to approximate, say, the area under the graph of a function and above a closed interval then become trapezoids. Simpson's method replaces the slanted-line tops with parabolas.

Though two points determine the equation of a line, three are required for a parabola. We also need to develop a formula for the area of a parabolic-top area-element if the sum of such areas is to become the Simpson approximation.



Suppose we consider a parabola $y = Ax^2 + Bx + C$ with its axis parallel to the y-axis and passing through three equally spaced points $(-h, y_L)$, $(0, y_M)$, and (h, y_R). Then substituting the three points into the equation gives three equations in the three unknowns A, B, C.

$$\begin{aligned} y_L &= Ah^2 - Bh + C \\ y_M &= C \\ y_R &= Ah^2 + Bh + C \end{aligned}$$

Solving these three equations by adding the first to the last, and then by subtracting the last from the first, yields:

$$\begin{aligned} 2Ah^2 &= y_L + y_R - 2y_M \\ B &= \frac{1}{h}\,\frac{y_R - y_L}{2} \\ C &= y_M \end{aligned}$$

Next, we compute the area under the parabola $y = Ax^2 + Bx + C$ and above the interval $[-h, h]$ for the values of A, B, and C we just found:

$$
\begin{aligned}
\int_{-h}^{h} Ax^2 + Bx + C\,dx &= \left(A\frac{x^3}{3} + B\frac{x^2}{2} + Cx\right)\Big|_{-h}^{h} \\
&= \frac{1}{3}\,2Ah^3 + 2Ch \\
&= h\left(\frac{1}{3}\,2Ah^2 + 2C\right) \\
&= h\left(\frac{1}{3}\left(y_L + y_R - 2y_M\right) + 2y_M\right) \\
&= \frac{h}{3}\left(y_L + y_R - 2y_M + 6y_M\right) \\
&= \frac{h}{3}\left(y_L + y_R + 4y_M\right)
\end{aligned}
$$

The above formula holds for the area of a parabolic topped area element with base of length $2h$ and vertical edges of length y_L on the left and y_R on the right. The height at the midpoint is y_M.

Now, let n be an even positive integer, and suppose we divide an interval $[a, b]$ into n equal parts each of length $h = \frac{b-a}{n}$. And suppose f is a function defined on $[a, b]$. As before we label the resulting partition $a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$, where for each j, $x_j = a + jh$, $0 \leq j \leq n$. And again, we let $y_j = f(x_j), 0 \leq j \leq n$. That is, the vertical edges go from the x-axis to the graph of f.

Next, start at the left endpoint a of the interval and erect a parabolic-top area-element on the first two subintervals. The base of this area-element goes from x_0 to x_2, and we use as vertical sides the lines that intersect the graph at (x_0, y_0) on the left and (x_2, y_2) on the right. The point (x_1, y_1) on the graph of f at the midpoint of the interval gives the third point we need to determine the parabola that forms the top of the area-element. From the formula we developed above, the area of this area-element is equal to $\frac{h}{3}\left(y_0 + y_2 + 4y_1\right)$.

If we repeat this process using the next two subintervals that go from x_2 to x_4, then the area of the resulting parabolic-top element will be (from an application of the formula above) $\frac{h}{3}\left(y_2 + y_4 + 4y_3\right)$. Thus, the sum of the areas of the two parabolic-top elements equals $\frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$. We continue in this way until we have calculated the areas of the $\frac{n}{2}$ parabolic-top area elements and added them together.

A pattern begins to emerge in the form of the sum of the areas of the $\frac{n}{2}$ parabolic-top area-elements. The sum will equal $\frac{h}{3}$ multiplied by: $y_0 + y_n$, i.e. the sum of the heights of the leftmost and rightmost vertical edges; plus 4 times the sum of the odd-indexed heights; plus 2 times the sum of the even-indexed heights because these edges belong to two successive area-elements, one on the left and the other on the right. This explains the form of the Simpson's Rule approximation which we now state

Definition: Let n be even. The n-subinterval Simpson approximation to $\int_a^b f(x)\,dx$ is given by

$$
\begin{aligned}
S_n &= \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right) \\
&= \frac{h}{3}\left(y_0 + y_n + 4\sum y_{odd} + 2\sum y_{even}\right)
\end{aligned}
$$

Example 3: Find S_4 for $\int_1^2 \frac{1}{x}\,dx$. The exact answer is $\ln 2$, or approximately 0.6931471806. In Example 1 we found that T_5 is equal to about 0.0696. If we are to use Simpson's rule for an approximation, then n has to be even. Therefore, S_4 is a legitimate sum to calculate. Note that $h = 1/4$. The five points of the partition are $x_0 = 1$, $x_1 = 5/4$, $x_2 = 3/2$, $x_3 = 7/4$, $x_4 = 2$. And the corresponding y-values are $y_0 = 1$,

$y_1 = 4/5$, $y_2 = 2/3$, $y_3 = 4/7$ and $y_4 = 1/2$. Thus,

$$\begin{aligned} S_4 &= \frac{1}{12}\left(1 + \frac{1}{2} + 4\,(y_1 + y_3) + 2\,(y_2)\right) \\ &= \frac{1}{12}\left(1 + \frac{1}{2} + 4\left(\frac{4}{5} + \frac{4}{7}\right) + 2\left(\frac{2}{3}\right)\right) \\ &\approx 0.6932539683. \end{aligned}$$

Note that S_4 with a smaller n is a better approximation to the actual value of the integral than T_5.

Example 4: Find S_4 for $\int_0^1 \sqrt{1 - x^2}\, dx$. The exact answer is $\pi/4$, or approximately 0.7853981635, one-quarter of the area of a circle of radius 1. In Example 2 we found that T_5 is equal to about 0.7592622072. If we are to use Simpson's rule for an approximation, then n has to be even, so S_4 makes sense. Note that $h = 1/4$. The five points of the partition are $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 3/4$, $x_4 = 1$. And the corresponding y-values are $y_0 = 1$, $y_1 = \sqrt{1 - 1/16}$, $y_2 = \sqrt{1 - 1/4}$, $y_3 = \sqrt{1 - 9/16}$ and $y_4 = 0$. Thus,

$$\begin{aligned} S_4 &= \frac{1}{12}\,(1 + 0 + 4\,(y_1 + y_3) + 2\,(y_2)) \\ &= \frac{1}{12}\left(1 + 0 + 4\left(\sqrt{15/16} + \sqrt{7/16}\right) + 2\sqrt{3/4}\right) \end{aligned}$$

or about 0.7708987887. The latter is a better approximation with a smaller n than we got with the Trapezoid rule.

Error Comparisons: As we found to be true in the examples, Simpson's rule is indeed much better than the Trapezoid rule. As $n \to \infty$ it generally converges much more rapidly to the value of the definite integral than does the Trapezoid rule.

We can get a sense of the differences in the rates of convergence of the two methods from the folowing two theorems:

Th1: Suppose the second derivative of f is continuous and hence necessarily bounded by a positine number M_2 on $[a, b]$. If $error_{T_n} = \int_a^b f(x)\, dx - T_n$, then

$$|error_{T_n}| \le \frac{M_2(b-a)^3}{12n^2}$$

Th2: Suppose the fourth derivative of f is continuous and hence necessarily bounded by a positive number M_4 on $[a, b]$. If $error_{S_n} = \int_a^b f(x)\, dx - S_n$, then

$$|error_{S_n}| \le \frac{M_4(b-a)^5}{180n^4}$$

These theorems imply that in many situations, as $n \to \infty$, $|error_{T_n}| \to 0$ like $1/n^2$ and $|error_{S_n}| \to 0$ like $1/n^4$. This explains why in general we are not surprised to find that Simpson's rule converges to the value of the integral much faster than the Trapezoid rule.

Importance of the Trapezoid and Simpson Rules: You might ask,What is the point of the Trap and Simp approximations in this age of computers? The answer is that they are simple to use and give excellent results, surprisingly so even for small n. A little arithmetic can yield a good estimate of a definite integral with only modest effort. Not bad, eh?

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