Techniques of Integration

In this section we are going to introduce the first approaches to evaluating an indefinite integral whose integrand does not have an immediate antiderivative. We begin with a list of integrals we should recognize.

$$\begin{aligned}
\int u^r\,du &= \frac{u^{r+1}}{r+1} + C, r \neq -1\\
\int \frac{1}{u}\,du &= \ln|u| + C\\
\int \sin u\,du &= -\cos u + C\\
\int \cos u\,du &= \sin u + C\\
\int \sec^2 u\,du &= \tan u + C\\
\int \sec u \tan u\,du &= \sec u + C\\
\int e^u\,du &= e^u + C
\end{aligned}$$

We can readily verify an equation by differentiating the right hand side and showing that we get the integrand on the left hand side. But what if an integral is not quite in the exact form that we require? For example, $\int e^{5x}\,dx$. What do we do then? Is there a systematic method that can minimize trial and error?

The Method of Substitution

We have already used differentials as an aid to integration when we discussed separable differential equations. In the present section, we will see that differentials continue to be a very useful technique for solving integrals. So, our first example will serve as a reminder of how to calculate them.

Example 1: If $y = x^3$, then $dy = 3x^2\,dx$. Or if $y = \sin 4x$, then $dy = 4\cos 4x\,dx$.

Reversing the Chain Rule: If $u = g(x)$ is a function of x, and f is a function of u, then the chain rule tells us that

$$(f(g(x)))' = f'(g(x))g'(x)$$

Thus, integrating the right hand side reverses the chain rule and we get

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + C$$

Now, we can rewrite the above integral by substituting into it $u = g(x)$ and the differential $du = g'(x)\,dx$. When we make these two substitutions we get

$$\int f'(u)\,du = f(u) + C$$

This last formula, combined with the use of differentials, constitutes the *Method of Substitution*.

Example 2: To find $\int e^{7x}\,dx$ by substitution, we look at the list of integrals at the beginning of the section and see that the *target* that it appears we should aim for is $\int e^u\,du$. Thus, we let $u = 7x$. Then we calculate $du = 7dx$; thus, $dx = \frac{du}{7}$. Substituting into the integral, we get the integral we were aiming for:

$$\int e^{7x}\,dx = \int \frac{e^u}{7}\,du = \frac{e^u}{7} + C = \frac{e^{7x}}{7} + C$$

We have already learned to solve the above integral by inspection. Indeed, we were doing nothing more than reversing the chain rule in an simple case. The next example is also one we have learned to do by inspection but which we can do formally by substitution.

Example 3: $\int \sin 2x\,dx$. Let $u = 2x$; then $du = 2dx$. So, substitution yields

$$\int \sin 2x\,dx = \int \frac{\sin u}{2}\,du = -\frac{\cos u}{2} + C = \frac{\cos 2x}{2} + C$$

Example 4: $\int \frac{x}{x^2+1}\,dx$. Let $u = x^2 + 1$; then $du = 2x\,dx$. So

$$\int \frac{x}{x^2+1}\,dx = \int \frac{1}{2u}\,du = \frac{1}{2}\ln|x^2+1| + C$$

Example 5: $\int \frac{x^2+1}{x}\,dx$. Be careful. This is not a substitution integral. That is, it is not an integral that requires substitution. We simplify the quotient to obtain two terms each of which we can integrate:

$$\int \frac{x^2+1}{x}\,dx = \int \left(x + \frac{1}{x}\right)\,dx = \frac{x^2}{2} + \ln|x| + C$$

Example 6: $\int \frac{x^2+1}{x^3+3x+2}\,dx$. This integral will yield to substitution: let $u = x^3 + 3x + 2$. Then $du = (3x^2+3)\,dx$. Thus, we substitute to get

$$\int \frac{x^2+1}{x^3+3x+2}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+3x+2| + C$$

Example 7: $\int \frac{\ln x}{x}\,dx$. Let $u = \ln x$; then $du = \frac{dx}{x}$. Thus,

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$$

Thus far, we have not used the Method of Substitution with a definite integral. We will do so now.

Example 7 (continued):

$$\int_e^{e^2} \frac{\ln x}{x}\,dx$$

This is our first example of a definite integral requiring substitution. There are two basic ways to solve it: either we change the variable from x to u and change the limits of integration as well; or we leave the limits of integration unchanged and switch back from u to x.

Method 1: change the limits from x to u.

x	$u = \ln x$
e	1
e^2	2

$$\int_e^{e^2} \frac{\ln x}{x}\, dx = \int_1^2 u\, du = \frac{u^2}{2}\bigg|_1^2 = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}$$

<u>Method 2</u>: change the variable back to x and retain the original limits.

$$\int_e^{e^2} \frac{\ln x}{x}\, dx = \frac{(\ln x)^2}{2}\bigg|_e^{e^2} = \frac{(\ln e^2)^2}{2} - \frac{(\ln e)^2}{2} = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}$$

Example 8:

$$\int_0^{\pi/4} \tan x\, dx$$

We replace $\tan x$ by $\tan x = \frac{\sin x}{\cos x}$ and then use the substituiton $u = \cos x$, from which $du = -\sin x\, dx$. So, with a change of the limits of integration

x	$u = \cos x$
0	1
$\pi/4$	$1/\sqrt{2}$

the integral becomes:

$$\int_0^{\pi/4} \tan x\, dx = \int_0^{\pi/4} \frac{\sin x}{\cos x}\, dx = \int_1^{1/\sqrt{2}} -\frac{1}{u}\, du = -\ln|u|\Big|_1^{1/\sqrt{2}} = \frac{\ln 2}{2}$$

Integration by Parts

We saw above that really the Method of Substitution consists of reversing the chain rule. Another technique of integration that is often useful involves an undoing of the product rule. For, suppose that u and v are functions of x. Then starting with the product rule we get

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$
$$\int u\frac{dv}{dx}\, dx = uv - \int v\frac{du}{dx}\, dx$$

Rewriting the last equation in terms of differentials yields

$$\int u\, dv = uv - \int v\, du$$

This is the so-called *integration by parts formula*. In practice, we can often use it to transform an integral that appears intractible into one whose integrand has an antiderivative we recognize.

Note that we first have to choose u; then dv is that part of the integrand that remains. In general, we must be able to differentiate u and integrate dv in order to use the rest of the parts formula. Also, we want an integrand that is simpler than the one with which we started. These simple observations should guide us in assigning u.

Let's consider some examples.

Example 1: Consider $\int xe^x\,dx$. Substitution does not appear to work. So, we try the only other technique we know, namely, parts. If we let $u = x$, then $dv = e^x\,dx$. Next, we find du and v, and use the parts formula.

$u = x$	$dv = e^x\,dx$
$du = dx$	$v = e^x$

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

Example 2: Given $\int \ln x\,dx$, we let $u = \ln x$ and then $dv = dx$. And we proceed:

$u = \ln x$	$dv = dx$
$du = \frac{1}{x}\,dx$	$v = x$

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C$$

Example 3: To find $\int_1^e \ln x\,dx$, we simply follow through in each term of the parts formula with the evaluation at the endpoints of the interval. Referring to the previous example,

$$\int_1^e \ln x\,dx = x\ln x|_1^e - \int_1^e dx = e - e + 1 = 1$$

Example 4: Given $\int x^2\sin x\,dx$, use parts letting $u = x^2$ and $dv = \sin x\,dx$. Then

$u = x^2$	$dv = \sin x\,dx$
$du = 2x\,dx$	$v = -\cos x$

$$\int x^2\sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$$

Now, we use parts again to evaluate the new integral.

$u = x$	$dv = \cos x\,dx$
$du = dx$	$v = \sin x$

$$\int x^2\sin x\,dx = -x^2\cos x + 2(x\sin x + \cos x) + C$$

Example 5: We use parts to evaluate $\int e^x\sin x\,dx$:

$u = e^x$	$dv = \sin x\,dx$
$du = e^x\,dx$	$v = -\cos x$

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

Using parts again:

$u = e^x$ $du = e^x \, dx$	$dv = \cos x \, dx$ $v = \sin x$

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

Now, we have an equation that has the unknown integral on both sides. Thus, we can solve for it to get

$$\int e^x \sin x \, dx = \frac{1}{2} \left(-e^x \cos x + e^x \sin x \right) + C$$

Exercises: Problems **Check what you have learned!**
Videos: Tutorial Solutions **See problems worked out!**