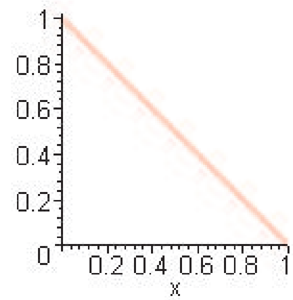
Properties of the Definite Integral

In the last section, we saw that if f is a nonnegative function on $[a, b]$, then the definite integral $\int_a^b f(x)dx$ is the area of the region under the graph of f and above the interval $[a, b]$. In fact, for most functions the definite integral *defines* the area under the graph. Before we consider some examples, let's give an obvious property of definite integrals but one worth noting.

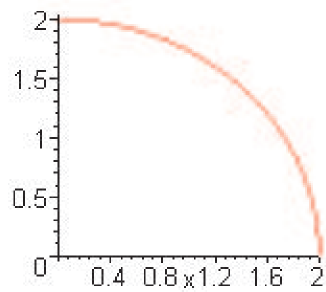
Property 1: $\int_a^a f(x)dx = 0$.

That is, if all of the Δx_i's are equal to 0, then the definite integral is 0. Now for some examples.

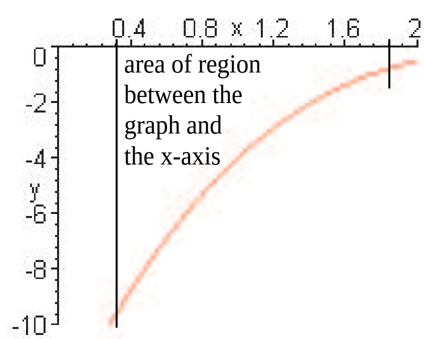
Example 1: Find $\int_0^1 (1 - x)dx$. From a sketch of the region, we see that the area is that of a right triangle whose legs are of length 1. Hence, the value of the integral is 1/2.



Example 2: Find $\int_0^2 \sqrt{4 - x^2}\, dx$. The area is that of a quarter-circle of radius 2 as shown in the sketch. Hence, from the area formula for a circle, we see that the value of the integral is $(\pi \cdot 2^2)/4 = \pi$.



If you look back at the definition of the definite integral, you will see that there is no requirement that f be a nonegative function. In fact, suppose f is continuous (and hence the definite integral exists) but f is strictly negative on $[a, b]$. Then in forming the the sums $\sum_{i=1}^n f(c_i)\Delta x_i$, note that $f(c_i)$ will be negative for all i and hence the sum will be negative. In fact, we can see that the definite integral will be the negative of the area of the region between the interval $[a, b]$ on the x-axis and the graph of f. (**Remember: areas are always nonnegative, but an integral may be negative.**) We have just proved a property of the definite integral.



Property 2: If f is integrable and $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\, dx$ equals the area of the region under

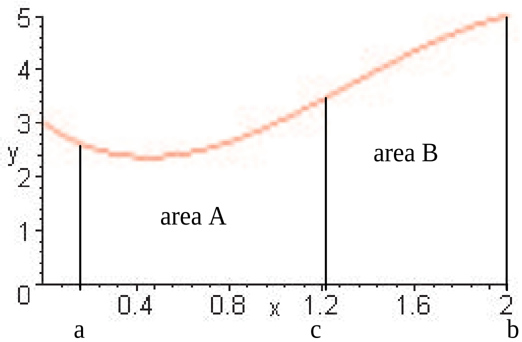
the graph of f and above the interval $[a,b]$. If $f(x) \le 0$ on $[a,b]$, then $\int_a^b f(x)\,dx$ equals the negative of the area of the region between the interval $[a,b]$ and the graph of f.

What happens if we integrate from right to left instead of from left to right? You guessed it.

Definition 1: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

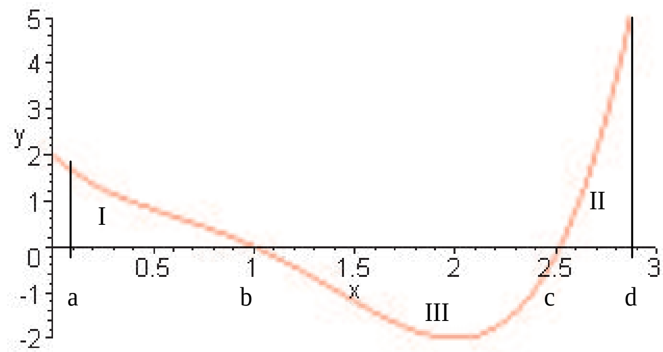
Because areas are additive, the following property also makes sense as seen in the sketch. That is, the total area over the interval $[a,b]$ is the sum of area A and area B. However, because of the above definition, the equation remains true even if c is not between a and b, or if b is less than a.

Property 3: $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$.



Consider now a function that is both positive and negative on $[a,b]$. Then because of Property 3 and Property 2, the value of the definite integral is the sum of the areas of the regions between $[a,b]$ and the graph of f *above* the x-axis, minus the sum of the areas of the regions between $[a,b]$ and the graph of f *below* the x-axis. Expressed more compactly, the definite integral is the sum of the areas above minus the sum of the areas below. (Conclusion: whereas area is always nonnegative, the definite integral may be positive, negative, or zero.)

Example 3: In the sketch below, if I, II, and III represent the areas (all positive numbers) of the depicted regions, then $\int_a^d f(x)\,dx =$ I+II-III.

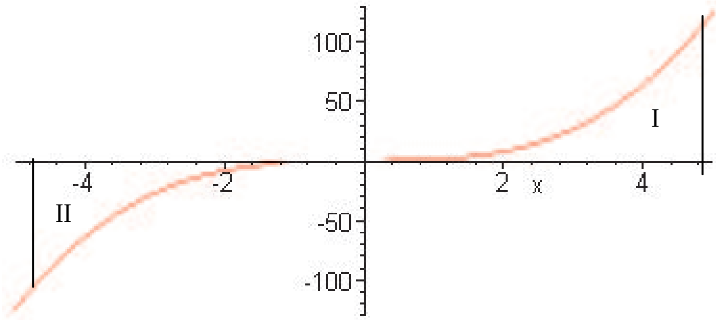


Integrals also add in the vertical direction. To understand this, note that for the finite sums that are involved in the definition of the integral, $\sum_{i=1}^n (Af(x_i) + Bg(x_i)) = \sum_{i=1}^n Af(x_i) + \sum_{i=1}^n Bg(x_i) = A\sum_{i=1}^n f(x_i) + B\sum_{i=1}^n g(x_i)$. As we pass to the limit in defining the integral, we need to use the fact that the limit of the sums is the sum of the limits. That is, we need to know that each limit exists. Hence we must assume that f and g are integrable to obtain the following result.

Property 4: If f and g are integrable on $[a,b]$, then $\int_a^b (Af(x)+Bg(x))\,dx = A\int_a^b f(x)\,dx + B\int_a^b g(x)\,dx$ for any constants A and B.

For some functions we can use symmetry to rewrite the integral in a simpler form. For instance, we have seen that odd functions are symmetric about the origin, and even functions are symmetric about the y-axis. Thus, we have the following two properties.

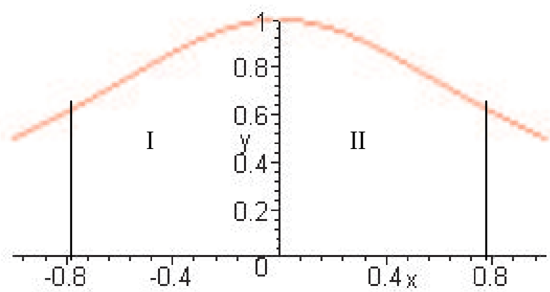
Property 5: If f is an odd function, then $\int_{-a}^a f(x)\,dx = 0$. That is, the definite integral of an odd function over a symmetric interval is zero.



area I = area II

In the case of even functions, the areas above and below the axis on a symmetric interval don't cancel, of course. However, we do have the following.

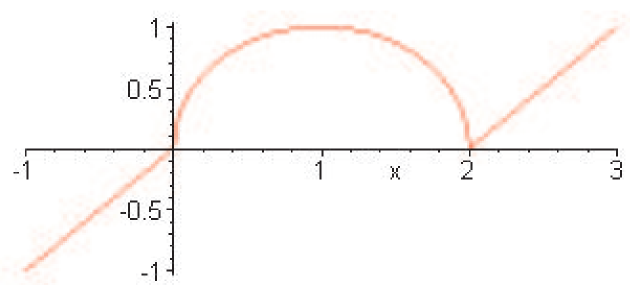
Property 6: If f is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.



area I = area II

Example 4: Let the function f be defined piecewise by

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ \sqrt{-x^2 + 2x} & 0 \leq x \leq 2 \\ x - 2 & \text{otherwise} \end{cases}$$



Then from what we know about the areas of triangles and circles, we have

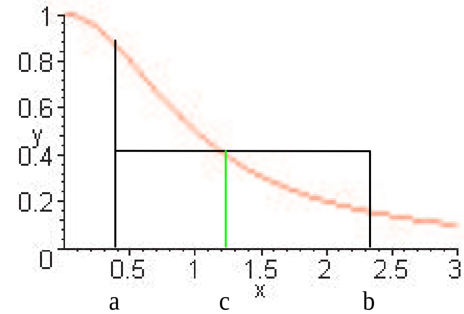
$$\int_{-1}^{3} f(x)\,dx = -\frac{1}{2} + \frac{\pi}{2} + \frac{1}{2} = \frac{\pi}{2}$$

Notice that $\int_{-1}^{0} f(x)\,dx = -\int_{2}^{3} f(x)\,dx$.

Mean Value Theorem for Definite Integrals

The next property is is a bit more subtle than the ones we have met so far. And yet it is easy to describe.

Consider the following sketch. It seems clear that we can find a rectangle with base $[a, b]$ whose area equals the area of the region under the graph of f and above $[a, b]$. We just draw the top edge of a rectangle and adjust its height until we find the rectangle for which the areas are equal. That is, the *extra* area of the rectangle equals the *omitted* area under the graph. The Mean Value Theorem for definite integrals says that this is always possible under not too restrictive conditions.



Theorem: (MVT for Definite Integrals) Let f be continuous on the interval $[a, b]$. Then there exists c in $[a, b]$ such that $\int_a^b f(x)\,dx = (b-a)f(c)$.

Note that on the right hand side of the equation we have the formula for the area of a rectangle of width $(b-a)$ and height $f(c)$. We think of $f(c)$ as the average value of f on the interval. Now, we can use the MVT formula to define *average value* precisely.

Definition 2: The average value of a continuous function on the interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

As soon as we have some relatively straightforward ways to compute definite integrals, we can return to this formula and make use of it.

Applet: Mean Value Theorem for Integrals **Try it!**

Exercises: Problems **Check what you have learned!**

Videos: Tutorial Solutions **See problems worked out!**