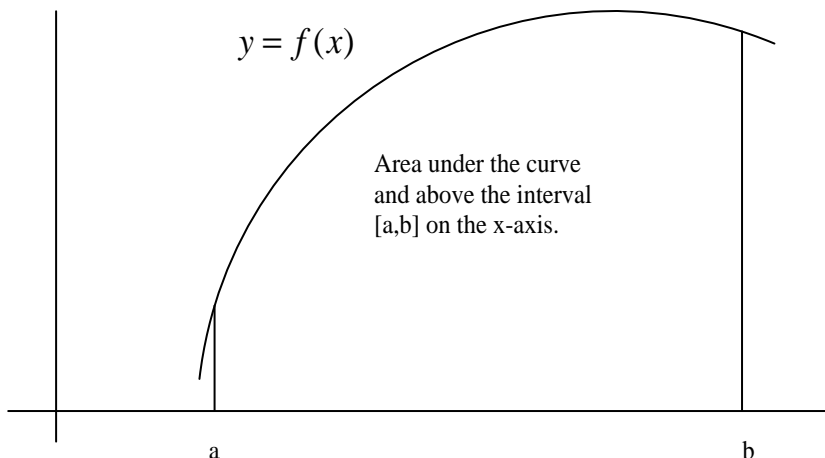


The Definite Integral

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Thus far, we have discussed the *Tangent Line Problem*. Its solution led to the definition of the derivative and to the rich array of applications that we have been studying. Now, we are ready to state the other fundamental problem with which calculus deals: *The Area Problem*.

The Area Problem: Find the area of the region in the xy -plane lying above the interval $[a, b]$ on the x -axis and under the graph of the nonnegative continuous function $y = f(x)$.



The problem seems approachable enough. That is, we are all familiar with areas and know how to calculate them for some basic geometric figures. In fact, before we go very far let's list three assumptions about areas that we can all agree to.

Assumptions about Areas:

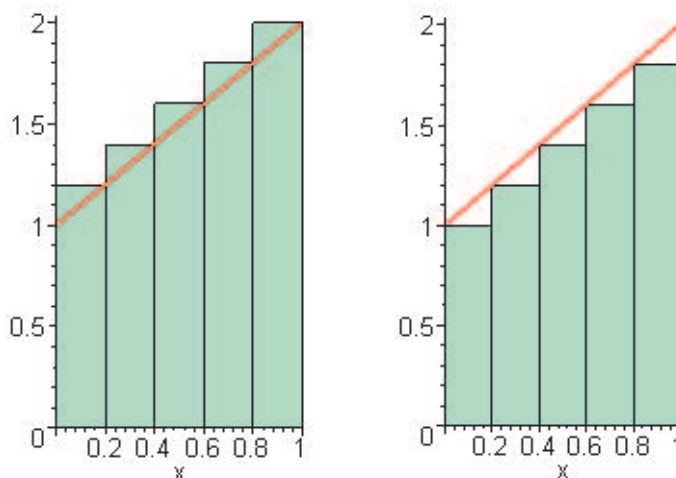
1. Area is a nonnegative number.
2. The area of a rectangle is its length times its width.
3. Area is additive. That is, if a region is completely divided into a finite number of non-overlapping subregions, then the area of the region is the sum of the areas of the subregions.

We probably do not need to say much about these assumptions. We have all computed the area of a square and a rectangle, and have subdivided regions into smaller regions whose areas we then added willy-nilly as the situation required to find the original area.

Returning to the Area Problem, because rectangles are convenient, our approach will be to use them to approximate the area of the region in question. We will start by considering an example and introducing some terminology.

Upper and Lower Sums; the Method of Exhaustion

Example 1: Suppose we want to use rectangles to approximate the area under the graph of $y = x + 1$ on the interval $[0, 1]$. Here are two possible ways to do it, as illustrated in the sketches.



In both sketches, we use five rectangles, but in the left picture, the area of the rectangle on each subinterval exceeds the area under the graph, while in the right picture the area of each rectangle is less than that of the corresponding subregion. The triangles at the top of each rectangle represent the amount by which we go over or fall short of the area of the region under the graph. We will call the sum of the areas of the rectangles in the left picture an *Upper Riemann Sum*, and the sum of the areas of the rectangles in the right picture a *Lower Riemann Sum*. Let's calculate these quantities.

Each rectangle is of width 0.2. In the Upper Sum, the height of each rectangle is f evaluated at the right endpoint of the subinterval; in the Lower Sum the heights are f evaluated at the left endpoint of the subinterval). Upper Sum = $.2f(.2) + .2f(.4) + .2f(.6) + .2f(.8) + .2f(1) = \frac{8}{5}$. Lower Sum = $.2f(0) + .2f(.2) + .2f(.4) + .2f(.6) + .2f(.8) = \frac{7}{5}$. Now, in the example we are considering, the region is the sum of a rectangle and a triangle. So, we know that the exact area is $1 + \frac{1}{2} = \frac{3}{2}$. And of course, $\frac{3}{2} = \frac{15}{10}$ is between $\frac{7}{5} = \frac{14}{10}$ and $\frac{8}{5} = \frac{16}{10}$.

Note that we can get a better approximation to the area under the graph by using rectangles of smaller width. For example, if we double the number of rectangles to 10 so that the width of each rectangle is 0.1, then the Upper Sum = $\frac{31}{20}$ and Lower Sum = $\frac{29}{20}$.

The process of increasing the number of rectangles to improve the approximation to the area whose value we seek is reminiscent of the *Greek Method of Exhaustion*. The inventors of calculus asked: Instead of stopping with a finite number of rectangles, why not take the limit of the sum of the areas of the rectangles as their widths approach 0? This should yield the exact value of the area, if (as always, an important proviso) the limit exists. Why not, indeed!

Let n stand for the number of rectangles, U for the Upper Riemann Sum, and L for the Lower Riemann Sum. Here are some values for the same example we have been discussing. You can compare these approximations with the exact value of 1.5:

n	U	L
100	1.505000000	1.495000000
150	1.503333333	1.496666667
200	1.502500000	1.497500000
300	1.501666667	1.498333333
500	1.501000000	1.499000000

General Procedure for finding the Area Under a Curve and Above an Interval: The above example suggests the following procedure for calculating the area under a curve.

1. Let $y = f(x)$ be given and defined on an interval $[a, b]$. Subdivide the interval $[a, b]$ into n subintervals. Label the endpoints of the subintervals $a = x_0 \leq x_1 \leq x_2 \leq x_3 \cdots \leq x_n = b$. Define

$P = \{x_0, x_1, x_3, \dots, x_n\}$ to be a *partition* of $[a, b]$.

2. Let $\Delta x_i = x_i - x_{i-1}$ be the width of the i^{th} subinterval, $1 \leq i \leq n$.
3. Form the Upper Riemann Sum $U(P, f)$: the height of each rectangle is the *maximum* value M_i of the function on that i^{th} subinterval.

$$U(P, f) = M_1 \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3 + \dots + M_n \Delta x_n$$

4. Form the Lower Riemann Sum $L(P, f)$: the height of each rectangle is the *minimum* value m_i of the function on that i^{th} subinterval.

$$L(P, f) = m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 + \dots + m_n \Delta x_n$$

5. Take the limit as $n \rightarrow \infty$ and the maximum $\Delta x_i \rightarrow 0$.

We have that $L(P, f) \leq \text{Area} \leq U(P, f)$. So, if the limit of the Upper Riemann Sums and the limit of the Lower Riemann Sums approach a common value, **this number is defined** to be the **area** under the curve and above the interval $[a, b]$.

Sigma Notation

From our discussion of the example above, we seem to have defined a working procedure to find the area of a region lying above an interval of the x -axis and under the graph of a function. But before going further, the process can be facilitated by introducing some useful notation.

Sigma Notation: If m and n are integers with $m \leq n$, and if f is a function defined on the integers from m to n , then the symbol $\sum_{i=m}^n f(i)$, called sigma notation, is defined to be $f(m) + f(m+1) + f(m+2) + \dots + f(n)$.

So, sigma notation is just a way of writing the sum in a compact form. (The word *sigma* comes from the Greek letter Σ .)

Example 2: Here are three examples:

1. $\sum_{i=1}^n i = 1 + 2 + 3 + 4 \dots + n$
2. $\sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + 4^2 \dots + n^2$
3. $\sum_{i=1}^n 1 = \underbrace{1 + 1 + 1 + 1 \dots + 1}_{n \text{ times}}$

Note that we can evaluate the sums in the above example by simply adding the numbers.

Example 3: If we add 1 to itself n times, the sum is n . So, $\sum_{i=1}^n 1 = n$. For instance, $\sum_{i=1}^5 1 = 1 + 1 + 1 + 1 + 1 = 5$.

Also,

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

For instance, $\sum_{i=1}^5 i = 1 + 2 + 3 + 4 + 5 = 15 = 5 \cdot 6 / 2$. This sum is often referred to as the *Gauss sum* because when he was a young school boy, the mathematical genius Gauss was able to solve the problem of adding the first 100 numbers for his teacher in lightning speed. Here is probably the way he did it:

1	2	3	4	5	...	98	99	100
100	99	98	97	96	...	3	2	1
101	101	101	101	101	...	101	101	101

That is, if you write the numbers first from 1 to 100, then in reverse order from 100 to 1, and add them, you get 100 times 101. But this is twice the answer you want, so you must divide by 2. Hence, the answer that you want is $(100 \cdot 101)/2 = 5050$. Pretty clever! Notice that there is nothing special about 100. We could prove the result we have stated for n in an analogous way, but we won't bother with the details here.

We will simply state the result for the sum of the squares of the first n numbers:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

Example 4:

$$\begin{aligned} \sum_{i=1}^{10} (3i^2 + 2i + 1) &= 3 \sum_{i=1}^{10} i^2 + 2 \sum_{i=1}^{10} i + \sum_{i=1}^{10} 1 \\ &= \frac{3 \cdot 10 \cdot 11 \cdot 21}{6} + \frac{2 \cdot 10 \cdot 11}{2} + 10 \\ &= 1275 \end{aligned}$$

The Area Problem Revisited

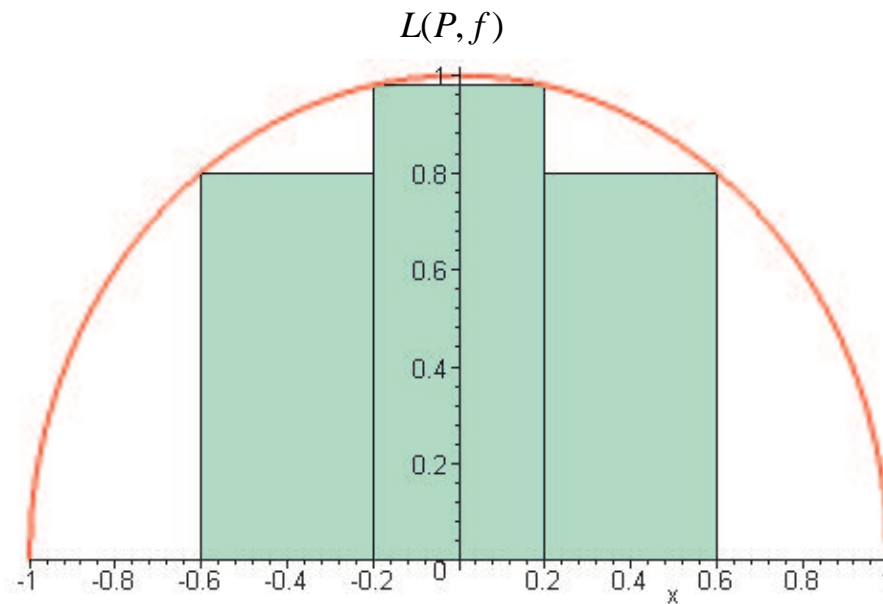
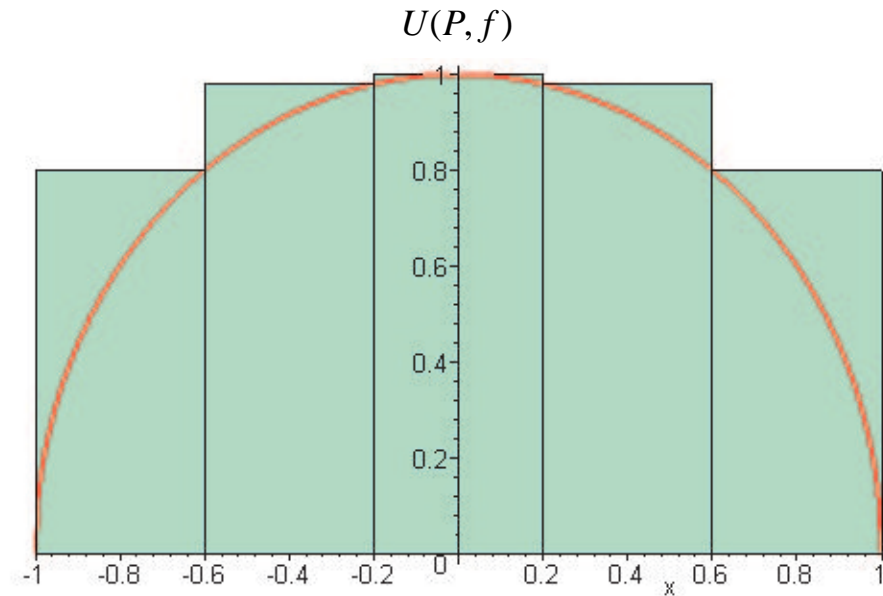
So far as an example we have considered a region whose top boundary is a line. And based on that example, we have outlined some fairly general procedures. Let's approximate the area of another region that we recognize just to be sure that we are going in the right direction. We will also make use of the terminology we have introduced. Note that in sigma notation the Upper and Lower Riemann sums can be stated compactly as

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i$$

$$L(P, f) = \sum_{i=1}^n m_i \Delta x_i$$

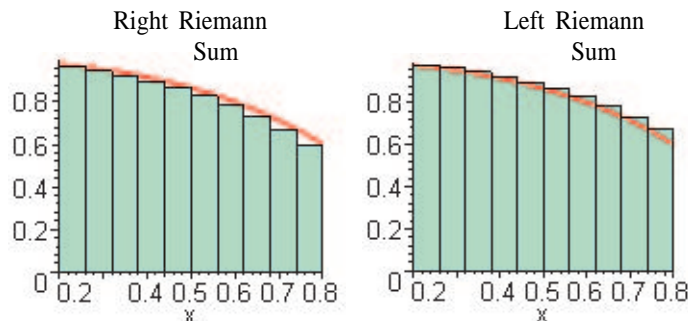
where M_i and m_i are, respectively, the maximum and minimum values of f on the i^{th} subinterval $[x_{i-1}, x_i]$, $1 \leq i \leq n$.

Example 5: Let $f(x) = \sqrt{1-x^2}$ on the interval $[-1, 1]$. Using 5 subintervals of equal width, find $U(P, f)$ and $L(P, f)$. To solve this problem, first note that the region is bounded by a semicircle and the x -axis. In this example, we are given that the widths of the rectangles are all the same, namely, $\Delta x = 2/5 = 0.4$. To form the Upper Riemann Sum, we use the maximum height of the rectangle on each subinterval. So, we get that $U(P, f) = \Delta x(f(-.6) + f(-.2) + f(0) + f(.2) + f(.6)) \approx .4(4.559591794) \approx 1.823836718$. The Lower Riemann Sum uses the minimum height of the rectangle on each subinterval. Thus, $L(P, f) = \Delta x(f(-.6) + f(-.2) + f(.6)) \approx .4(2.579795897) \approx 1.031918359$. The actual value of the area is $(\pi \cdot 1^2)/2$ which is approximately 1.570796327. Once again, we would expect that as we let $\Delta x \rightarrow 0$, we would get a better and better approximation of the area under the graph.



There are two other Riemann Sums that are convenient to use because their formulas do not depend on the characteristics of the function. Given a partition P of $[a, b]$, $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$, and $\Delta x_i = x_i - x_{i-1}$ the width of the i^{th} subinterval, $1 \leq i \leq n$; let f be defined on $[a, b]$. Then the **Right Riemann Sum** is $\sum_{i=1}^n f(x_i)\Delta x_i$ and the **Left Riemann Sum** is $\sum_{i=0}^{n-1} f(x_i)\Delta x_i$

The left Riemann Sum uses the left endpoint of each subinterval to determine the height of the rectangle on that subinterval, while the Right Riemann Sum uses the right endpoint. Note that if f is decreasing on $[a, b]$, then $U(P, f)$ is a Left Riemann Sum, and $L(P, f)$ is a Right Riemann Sum. Similar comments apply to increasing functions.



Example 6: Approximate the area of the region under the graph of the function $f(x) = \sqrt{1-x^2}$ on the interval $[0,1]$ using a Left and a Right Riemann Sum first with 5 rectangles of equal width, then with 100. Note that because the function that describes the quarter-circle is decreasing on $[0,1]$, the question asks us to find the Upper and Lower Riemann Sums for 5, and then 100 subintervals of equal width. A big advantage to the left and right Riemann sums is that their formulas are easily programmed into a programmable calculator or a computer. In this example, in the case of 5 rectangles, $x_i = 0 + i/5, 0 \leq i \leq 5$, and we want to find the Left Riemann Sum

$$\frac{1}{5} \sum_{i=0}^4 \sqrt{1-x_i^2} = \frac{1}{5} \sum_{i=0}^4 \sqrt{1-\frac{i^2}{25}}$$

and then the Right Riemann Sum

$$\frac{1}{5} \sum_{i=1}^5 \sqrt{1-x_i^2} = \frac{1}{5} \sum_{i=1}^5 \sqrt{1-\frac{i^2}{25}}$$

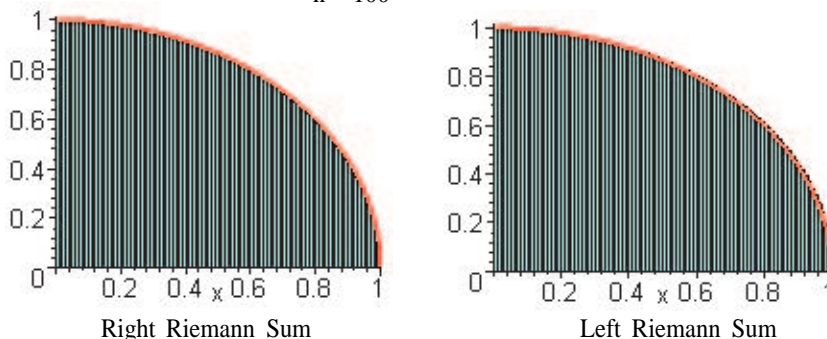
These evaluate to Left Riemann Sum $\approx .8592622072$ and Right Riemann Sum $\approx .6592622072$. The actual value is $\pi/4 \approx .7853981635$.

With 100 subintervals, we can get even closer to $\pi/4$ by evaluating the sums

$$\frac{1}{100} \sum_{i=0}^{99} \sqrt{1-x_i^2} = \frac{1}{100} \sum_{i=0}^{99} \sqrt{1-\frac{i^2}{100^2}} \approx .7901042579$$

$$\frac{1}{100} \sum_{i=1}^{100} \sqrt{1-x_i^2} = \frac{1}{100} \sum_{i=1}^{100} \sqrt{1-\frac{i^2}{100^2}} \approx .7801042577$$

$n = 100$



Applet: [Riemann Sums](#) Try it!

The Definite Integral

Believe it or not, we almost have the definition of the definite integral in hand. We will state it formally so that we can refer to it conveniently as needed.

Definition: Let P be a partition of the interval $[a, b]$, $P = \{x_0, x_1, x_2, \dots, x_n\}$ with $a = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n = b$. Let $\Delta x_i = x_i - x_{i-1}$ be the width of the i^{th} subinterval, $1 \leq i \leq n$. Let f be a function defined on $[a, b]$. Next form the Upper Riemann Sum $U(P, f)$ where the height of the rectangle on each subinterval is the maximum value of f on that subinterval; and form the Lower Riemann Sum $L(P, f)$, where the height of the rectangle on each subinterval is the minimum height of f on that subinterval. Then we say that f is Riemann integrable on $[a, b]$ if there exists a unique number Φ such that $L(P, f) \leq \Phi \leq U(P, f)$ for all partitions of $[a, b]$. We write the number Φ as

$$\Phi = \int_a^b f(x)dx$$

and call it the definite integral of f over $[a, b]$.

The integral symbol is a stylized Greek sigma Σ from the summation notation we introduced above. The x is a so-called *dummy* variable in that it merely tells us the variable with respect to which we are integrating; hence, we could equally well write $\int_a^b f(t)dt$ or $\int_a^b f(r)dr$.

The definition looks a bit awkward to verify. However, there are two important theorems that come to our aid from advanced analysis, and which we rely on in practice.

Theorem 1: If f is Riemann integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{\substack{n \rightarrow \infty \\ \|P\| \rightarrow 0}} \sum_{i=1}^n f(c_i)\Delta x_i$$

where c_i is any point in the subinterval $[x_{i-1}, x_i]$, and $\|P\|$ is the maximum length of the Δx_i .

So, the Upper Riemann Sum, the Lower Riemann Sum, the Left Riemann Sum, and the Right Riemann Sum are all special cases of the sum in the above limit where we choose the points c_i in very particular ways. (That is, where f is a maximum, or a minimum, or the left endpoint, or the right endpoint, respectively.) In the examples, we usually take the subintervals to be of equal length, so as $n \rightarrow \infty$, the length of each subinterval automatically goes to 0.

The above theorem is much more than a theoretical result. We will see that we use it extensively in applications as a guide in setting up a mathematical model connected with the problem. But more about that later. The theorem below allows us to work effectively with the integral because most of the functions in which we will be interested are continuous or piecewise continuous.

Theorem 2: If f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.

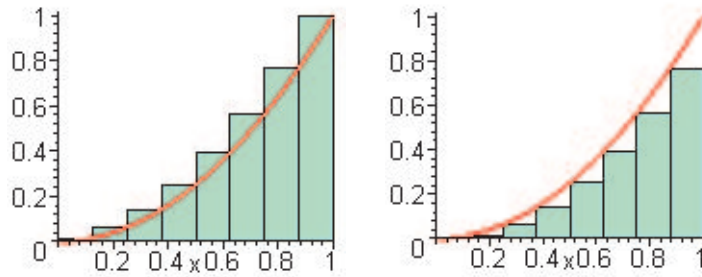
This theorem tells us that for continuous functions, we can use the limit of any convenient Riemann sums to evaluate the integral.

Example 7: Use an Upper Riemann Sum and a Lower Riemann Sum, first with 8, then with 100 subintervals of equal length to approximate the area under the graph of $y = f(x) = x^2$ on the interval $[0, 1]$.

First with 8 subintervals:

$$U(P, f) = \frac{1}{8} \sum_{i=1}^8 \frac{i^2}{64} \approx .3984375$$

$$L(P, f) = \frac{1}{8} \sum_{i=0}^7 \frac{i^2}{64} \approx .2734375$$



Then with 100 subintervals:

$$U(P, f) = \frac{1}{100} \sum_{i=1}^8 \frac{i^2}{10000} = .33835$$

$$L(P, f) = \frac{1}{100} \sum_{i=0}^7 \frac{i^2}{10000} = .32835$$

Exercises: [Problems](#) Check what you have learned!
Videos: [Tutorial Solutions](#) See problems worked out!