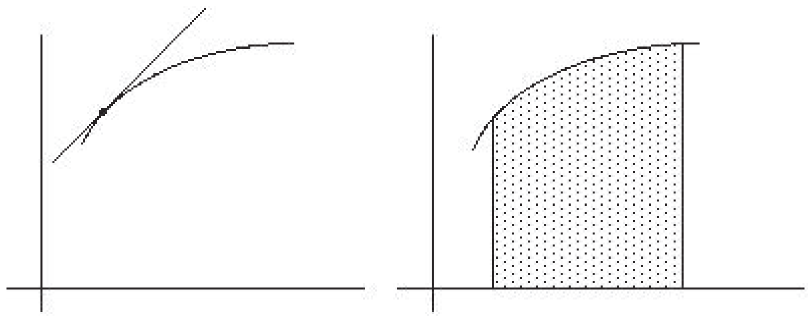
Modeling Accumulations: Introduction to the Issues

Traditionally, the purpose of calculus is twofold: to find the slope of a curve at a point; and to find the area lying under a curve and above an interval of the x-axis.



We have already dealt with the first problem. Its solution leads to the definition of the derivative. The derivative of a function at a point is then the slope of the tangent line to the graph of the function at the point. Moreover, the original issue of studying the slope of a curve gets transformed into the much more general issue of defining the rate of change of a function. What began as a somewhat restrictive investigation and set of concerns explodes into a set of tools for addressing very general problems in dynamic settings limited only by one's imagination.

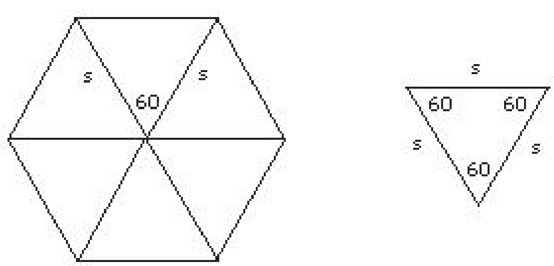
The second concern of calculus, that of finding the area under a curve, will also turn out to have very far reaching consequences. But before discussing the generalities, we will begin with some examples of the area problem itself. In this way, we will become familiar with what is at issue, and we will be able to establish an agenda for future work.

0.1 The Area of a Circle

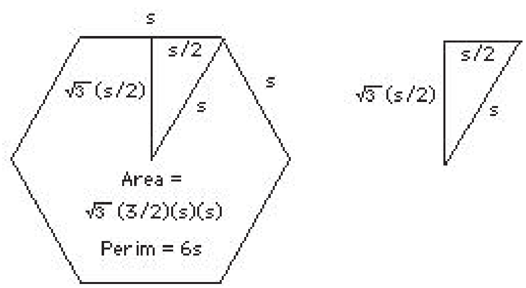
We all know the formula A for the area of a circle: $A = \pi r^2$, where r is the radius and π is the irrational number whose decimal expansion to 20 decimal places begins 3.14159265358979323846. But have you ever stopped to wonder what this all means? That is, what exactly *is* the area of a circle? What is its *definition*?

The last question is not easy to answer, is it? Think about the question a bit just to see where you come out. For now, we will postpone an answer and turn to a question we can answer fairly straightforwardly, namely, how do we compute the area of a circle? For circles of radius 1, this is equivalent to asking, how do we compute π?

Consider a circle whose radius is of length one, a so-called *unit circle*. Our task is to compute its area. Archimedes faced this same problem centuries ago, and his methods are still valid today. The trick is to approximate the area of the circle by that of a geometric figure whose area can be calculated from a simple formula (not the kind that involves a number like π). A hexagon turns out to be an excellent starting place for these purposes.



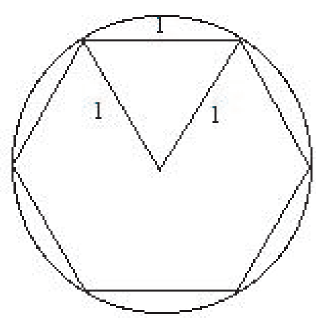
Pictured above is a regular hexagon. Notice that it is composed of six congruent isoseles triangles, each with a 60 (= 360/6) degree central angle. Thus, the base angles of each triangle are also 60 degrees, and the third side has the same length as the other two. Hence, we can find the height of each isoseles triangle and its area using the relationship between the lengths of the legs of a 30-60-90 degree right triangle. With reference to the sketch below, the area of one of the isoseles triangles is $\frac{1}{2}s\frac{\sqrt{3}s}{2}$.



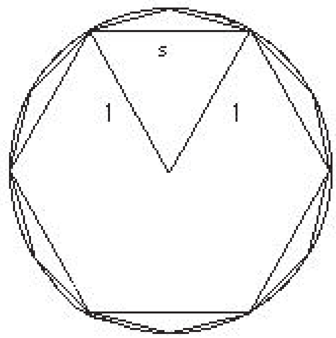
Thus, the area of the hexagon is six times as large, or

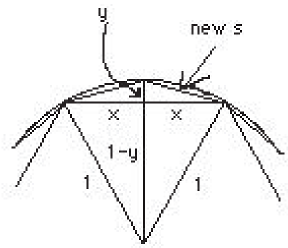
$$\frac{3}{2}\sqrt{3}s^2$$

Let's not lose sight of our objective: We want to find the area of a unit circle, and hence the value of π. As a first approximation, we will use the area of an inscribed hexagon. That is, with $s = 1$, the formula we have just derived tells us that the area of the hexagon is $\frac{3\sqrt{3}}{2}$, or approximately 2.598.



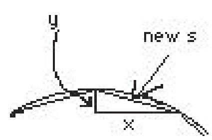
This does not give a very good approximation to the area of the circle. We surely can do better, but how? We could replace the hexagon with more completely filling shapes whose areas we can still calculate. One of the best ways to do this is to double the number of edges of the hexagon, thereby obtaining a regular 12-gon; and then to continue doubling repeatedly to obtain in succession a 24-gon, a 48-gon, a 96-gon, and so on. This certainly makes sense from the viewpoint of filling the area of the circle. It also yields figures whose areas can be calculated readily from one stage to the next.





Let's calculate the area of the 12-gon. In the sketch above, $x = s/2$ and from the Pythagorean Theorem we find that $x^2 + (1-y)^2 = 1$. Solving for y in just a few steps yields $(1-y)^2 = 1 - x^2$, $1 - y = \sqrt{1-x^2}$, or $y = 1 - \sqrt{1-x^2}$. Also, $(new\ s)^2 = x^2 + y^2$, or $new\ s = \sqrt{x^2+y^2}$. Thus, the area of the 12-gon equals the area of the hexagon plus 12 times the area of a little triangle:

$$\frac{3}{2}\sqrt{3} + 12\frac{xy}{2}$$



In fact, we can double the number of sides from 12 to 24 by repeating the steps here with *new s* replacing s and x_1 and y_1 replacing x and y, respectively. The area of the 24-gon is then

$$\frac{3}{2}\sqrt{3} + 12\frac{xy}{2} + 24\frac{x_1y_1}{2}$$

Summarizing, if we begin with a hexagon inscribed in a circle of radius 1, and obtain a sequence of regular polygons by doubling the number n of sides, then at each stage we obtain a new approximation to the area of the circle by completing the following five steps:

1. $x = s/2$ [To begin $s = 1$.]
2. $y = 1 - \sqrt{1-x^2}$
3. $new\ s = \sqrt{x^2+y^2}$
4. $new\ n = 2n$ [To begin $n = 6$.]
5. $new\ A = A + (new\ n)\frac{xy}{2}$ [To begin $A = \frac{3}{2}\sqrt{3}$.]

Here is a table showing the results of 10 doublings.

Areas of Regular Polygons	
sides	area
6	2.598076
12	3.000000
24	3.105829
48	3.132629
96	3.139350
192	3.141032
384	3.141452
768	3.141558
1536	3.141584
3072	3.141590
6144	3.141592

Thus, we get an approximation to the area of a unit circle, and hence an approximation to π. The best approximation in the table comes from a regular inscribed polygon of 6144 sides. Clearly, we could do even better by doubling the number of sides to 12288, 24576, etc.

Applet: Approximating Areas: Inscribed Polygons **Try it!**

0.2 What is the Area of a Circle?

Returning to the earlier question that we postponed during our calculations, do we now know what the area of a unit circle is? You may answer: Of course, it is π. But π is the *value* of the area of a unit circle. We seem to be in the position of knowing the value of something without knowing how to define that something. Note that we don't have this problem for the regular polygons that we have used. Beginning by defining the area of a triangle to be one-half the product of its height and the length of its base, we can define the area of a regular polygon to be the sums of the areas of its component triangles. This is precisely how we calculated the areas of the triangles and polygons above.

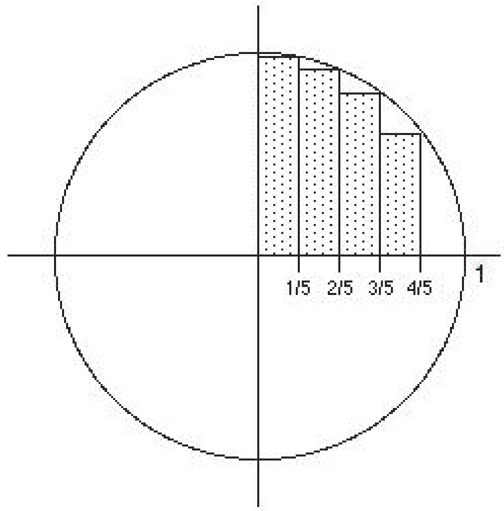
The definition of the area of a circle is not so simple. From what we have done so far, it seems to involve a more elaborate process. In fact, from our work it is reasonable to *define* the area of a unit circle to be the limit of the areas of the inscribed regular polygons that come from starting with a hexagon and doubling the number of sides at each successive stage.

Hold on now, you may say. Do you mean that the area of a circle is tied to a hexagon and the polygons that come from a doubling process? Said another way: What is so special about these geometrical figures? Why not use any collection of shapes that are contained in the circle and fill it in the limit?

0.3 Another Calculation of the Area of a Circle

If the questions in the last paragraph could have come from you, you certainly would be justified in your skepticism. Our decision to use an inscribed hexagon and a doubling process was based primarily on two factors: the polygons appear to fill the circle very rapidly; and their areas are relatively easy to calculate. However, we will see that other approaches are just as appealing, perhaps even more so.

For example, suppose we consider approximating the area of a quarter-circle with rectangles, as shown in the sketch. We divide the interval [0, 1] into n subintervals of equal length $h = 1/n$; in the sketch, $h = 1/5$. The circle is the graph of the function $f(x) = \sqrt{1 - x^2}$, where x is between 0 and 1, and the upper right-hand corner of each rectangle lies on the circle. Thus, the area of the rectangle on the subinterval, say, $[3/n, 4/n]$ is $(1/n)f(4/n)$. We then add the areas of the rectangles and use this as an approximation to the area of the quarter-circle. Note that the rectangle on the last subinterval is of zero height, and hence its area is 0.



In the sketch, there are 4 rectangles, and the sum of their areas is $(1/5)[f(1/5)+f(2/5)+f(3/5)+f(4/5)]$. We can increase the number of rectangles, and calculate the value of the sum of the areas of the rectangles for subintervals of shorter and shorter length; that is, for rectangles of narrower and narrower width. As the rectangles get narrower, they come closer to filling the quarter-circle. We will multiply the sums by 4 to obtain an approximation to the area of the full circle. Here are some results.

Approximating Area of Unit Circle with Rectangles

rectangles	sum of areas times 4
5	2.637049
500	3.137487
1000	3.139555
2000	3.140580
5000	3.141189

With just five rectangles we obtain a rather crude approximation of the area, but as the number of rectangles increases we begin to see the familiar value of π emerge. Certainly the polygon approximations deliver a better approximation to the area with fewer computational steps. But it appears that we could calculate the area of a circle as accurately as we please using a sufficient number of rectangles.

Applet: Approximating Area: Using Rectangles **Try it!**

Although the two methods used to approximate the area of a circle differ, it still seems clear that we could define the area in terms of a limiting process. Whether the limiting process is that of calculating the areas of inscribed polygons with an ever increasing number of sides, or calculating the sum of areas of inscribed rectangles as the width of the rectangles approach zero, the area of the circle can be defined in terms of a circle-filling limiting process involving simple geometrical figures. For calculating purposes it is important only that the areas of the geometrical figures are known and easy to calculate. This is certainly true for the rectangles. Their areas are readily computed because their heights are obtained immediately from the function whose graph is the circle. Let's make note of two central features we have identified in calculating areas and come back to them later: the first is *limit*, the second is *function*.

0.4 The Method of Accumulations

We started with the problem of finding the area under a curve and above an interval. Generalizing from the two approaches we have taken to finding the area of a circle, a fruitful approach seems to be to accumulate small pieces that approximate the area, and pass to the limit. In the limit, the area is filled, and hopefully we can then evaluate it as the limit of the values of the areas of the pieces. The process of *passing to the limit* not only provides a calculational tool, but it gives a way to *define* what is meant by the *area under the curve*. Even in the case of a figure as familiar as a circle, this in and of itself is a worthwhile accomplishment.

But area is not the only thing that lends itself to what we will call the *method of accumulations*. For example, suppose instead of the area of a circle we wanted to find the circumference.

0.5 The Circumference of a Circle

Archimedes showed that π is between 223/71 and 22/7. He did this by calculating the perimeters of the 96-sided regular polygons inscribed in, and circumscribed about, a circle. Because the circumference c of the circle lies between these two perimeters, and because $c = 2\pi r$, we get an estimate for π.

We can take Archimedes' hint and use the method of accumulations to find the circumference of a unit circle from the perimeters of inscribed polygons. In fact, we have already done most of the work when we found the area by starting with an inscribed hexagon and successively doubling the number of sides. Using our previous notation, *new* n is the number of sides of the polygon at the next stage, and *new* s is the length of a side. Thus, the improved estimate of the circumference is $(new\ s)(new\ n)$, and of π is $(new\ s)(new\ n)/2$. Here are the results for 10 doublings.

Circumferences of Regular Polygons

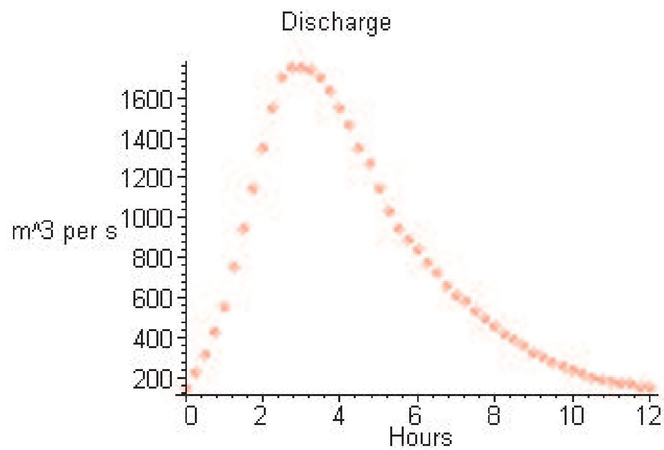
sides	perimeter	π
6	6.000000	3.000000
12	6.211656	3.105829
24	6.265260	3.132629
48	6.278700	3.139350
96	6.282066	3.141032
192	6.282906	3.141452
384	6.283116	3.141558
768	6.283170	3.141584
1536	6.283182	3.141590
3072	6.283182	3.141592
6144	6.283182	3.141593

So, by adding up the lengths of the sides of each polygon and letting the length of the sides get uniformly shorter, we get a progressively better approximation of the circumference of the circle. Again, the quantity we want to calculate (or define) is approached through a limiting process.

0.6 The Volume of Water in a River

Even though our examples thus far have been geometric in nature, the method of accumulations is really quite general. Its use often arises in the study of real-world applications. To illustrate this point, let us consider an example from the everyday world of flood forecasting. Suppose we measure over time the flow rate of a river stream. That is, we record at a finite number of times, the number of cubic meters of water that pass a fixed point during one second. Earth Scientists call the volume of water per unit time the *discharge* of a stream; mathematicians would prefer a term like *flow rate*. This is just one example of many in which specialists from different areas use different terminology for the same concept, and may even give different meanings to the same terms. However, as long as we understand both languages, there should be no problem talking to both groups. Since we are describing a problem in earth sciences, we will use their terminology. This also happens to be the terminology used to record the data that you might want to look up in a book or on the World Wide Web for a river of interest.

Prior to a rain storm, the stream will be flowing at some background level of discharge known as *base flow*. However, following a period of heavy precipitation, the rain falling in the watershed drains into the stream, and the discharge increases over time. The discharge of a stream does not rise immediately with the onset of precipitation, rather it takes time to flow across the watershed and into the stream. If the discharge of the stream exceeds the carrying capacity of the channel, the stream overflows its banks and floods. Here is an example of a typical table of discharge data for a 12-hour period. First the graph.

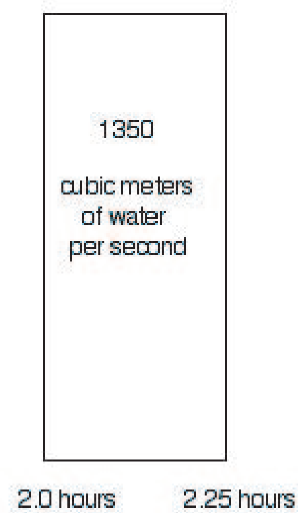


Then the table.

Discharge of a River Stream					
hours	m^3/s	hours	m^3/s	hours	m^3/s
0	150	4	1550	8	460
0.25	230	4.25	1460	8.25	423
0.5	310	4.5	1350	8.5	390
0.75	430	4.75	1270	8.75	365
1	550	5	1150	9	325
1.25	750	5.25	1030	9.25	300
1.5	950	5.5	950	9.5	280
1.75	1150	5.75	892	9.75	260
2	1350	6	837	10	233
2.25	1550	6.25	770	10.25	220
2.5	1700	6.5	725	10.5	199
2.75	1745	6.75	658	10.75	188
3	1750	7	610	11	180
3.25	1740	7.25	579	11.25	175
3.5	1700	7.5	535	11.5	168
3.75	1630	7.75	500	11.75	155
				12	150

There are many questions that earth scientists and regional planners may want to ask about the data and the stream. We will take these up in a more complete form later after we have some additional mathematical tools with which to work. For now, we will limit ourselves to answering one question: What is the volume of water that flowed past the fixed point in the stream during the 12 hours of recorded data?

To answer the question, we will first assume that the discharge is constant over each of the subintervals of time. The sketch below shows the situation for the interval of time from 2 to 2.25 hours, where 1350 cubic meters per second is taken from the table.



The volume of water that flows past the fixed point during this time interval is 1350 (cubic meters per second) times 3600 (seconds per hour) times 0.25 (hours); or, 1,215,000 cubic meters. To get the total volume over the 12 hours, we first find the volume over each subinterval and then, once again, add them together. Here is what we get.

Volume Over Each Subinterval					
subint	m^3	subint	m^3	subint	m^3
0	135000	4	1395000	8	414000
0.25	207000	4.25	1314000	8.25	380700
0.5	279000	4.5	1215000	8.5	351000
0.75	387000	4.75	1143000	8.75	328500
1	495000	5	1035000	9	292500
1.25	675000	5.25	927000	9.25	270000
1.5	855000	5.5	855000	9.5	252000
1.75	1035000	5.75	802800	9.75	234000
2	1215000	6	753300	10	209700
2.25	1395000	6.25	693000	10.25	198000
2.5	1530000	6.5	652500	10.5	179100
2.75	1570500	6.75	592200	10.75	169200
3	1575000	7	549000	11	162000
3.25	1566000	7.25	521100	11.25	157500
3.5	1530000	7.5	481500	11.5	151200
3.75	1467000	7.75	450000	11.75	139500
				12	135000

The sum of the volumes over the subintervals yields the total volume during the twelve hours: 33.1848 million cubic meters of water have passed the given point.

Once again, as we found in our previous examples, taking measurements closer together should yield a more accurate approximation of the total volume. On the other hand, given the nature of the problem, this probably is not necessary. After all, recording the data 15 minutes apart seems demanding enough on earth science personnel as it is.

We want to keep in mind how the example of river flooding differs from the ones we have looked at heretofore. First, it involves real data. Earth scientists really do collect discharge data and they really do use it to make predictions about flooding. Second, the sum, over a period of time, of rates-times-time quantities yields an approximation to the total thing (in this case volume) whose rate of change we have measured. The latter is a key point that we will return to later.

Applet: Accumulation: River Flow **Try it!**
Applet: Accumulation: Distance Traveled **Try it!**

0.7 Our Agenda for This Chapter

Calculus has two general lines of development:

1. slope → rate of change → derivative

2. area → method of accumulations → integration

We have discussed the derivative in the last section. In the present section we study the integral. At this point we only know that integration has something to do with the method of accumulation. So, as we plan our agenda for the Chapter, we will begin with a precise definition of the integral and proceed from there. In actuality, there are two but related notions of integral, the so-called *definite integral* and the *indefinite integral* that we already have defined as the *general antiderivative* of a function. Here is what we need to do to understand both integrals.

1. Develop an explicit definition of the definite integral.

2. Study the theoretical properties of integrals; in particular, relate the definite integral to area and accumulation.

3. Develop algebraic rules for finding integrals.

4. Develop numerical techniques for evaluating definite integrals.

5. Surprise: Discover the relationship between derivatives and integrals.

It is hard to motivate the last point in the list until we get more experience with integrals. However, Leibniz is the one who first drew special attention to the connection. It is so important that we call it *The Fundamental Theorem of Calculus*. This theorem makes a beautiful observation that unifies the conceptual framework of the study of calculus by relating derivatives and integrals to each other. Terrific!
Exercises: Problems **Check what you have learned!**
Videos: Tutorial Solutions **See problems worked out!**